Symmetries of the stationary Euler equations in the frame of dual stream function representation

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functions against each other at the nodes of the cell. An example *fg* diagram for a tetrahedral cell is shown in Fig.3. Note that both stream functions along a streamline are constant and streamlines are therefore reduced to points on the *fg* diagram.

There are two approaches to computing the stream functions: a global method and a local method. Both algorithms for computing stream functions on tetrahedral meshes are outlined below; the hexahedral case can be found in [Kenwright, 1992].

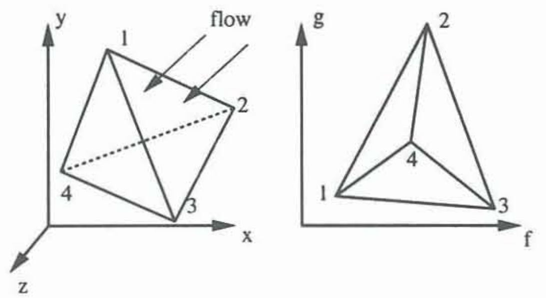


Figure 3: THE TRANSLATION FROM CARTESIAN TO *fg* SPACE FOR A GENERAL TETRAHEDRON.

Whole Field Solution Method Given the values of the dual stream functions at three nodes of a tetrahedron, the values at the fourth can be computed easily from the mass flux data. This is because the fourth node can be seen as a barycentric combination of the other nodes; the barycentric coordinates being computed from the relative fluxes through the faces of the tetrahedron:

$$f_4 = -\frac{\dot{m}_1}{\dot{m}_4}f_1 - \frac{\dot{m}_2}{\dot{m}_4}f_2 - \frac{\dot{m}_3}{\dot{m}_4}f_3 \quad (13)$$

$$g_4 = -\frac{\dot{m}_1}{\dot{m}_4}g_1 - \frac{\dot{m}_2}{\dot{m}_4}g_2 - \frac{\dot{m}_3}{\dot{m}_4}g_3 \quad (14)$$

provided $\dot{m}_4 \neq 0$. If the flux through the face corresponding to the unknown node is zero, a solution for both stream functions cannot be found, and the tetrahedron is skipped. It is usually possible to find the unknown stream functions from the *fg* diagrams of neighbouring tetrahedra. We can then travel through the mesh in a recursive fashion, computing the dual stream functions as we go.

This approach fails should either of the stream functions become multi-valued, as in areas of recirculating or spiralling flow. The stream surfaces can be visualised by constructing iso-surfaces, and streamlines obtained by calculating the intersection of iso-surfaces of both stream functions, as seen in Fig.4.

Local Solution Method If a whole field solution cannot be found, then a streamline can be computed by tracking through the mesh, computing the dual stream functions for cells as and when they are needed. The algorithm proceeds as follows:

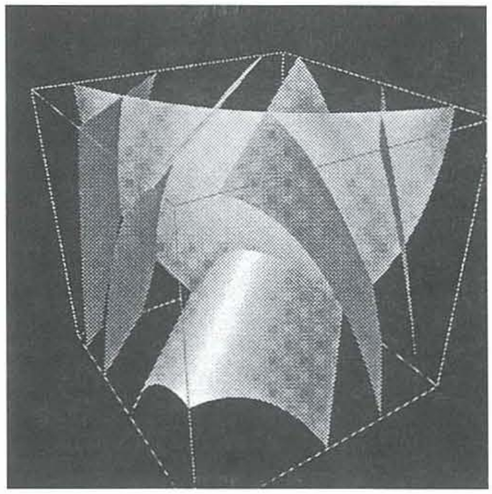
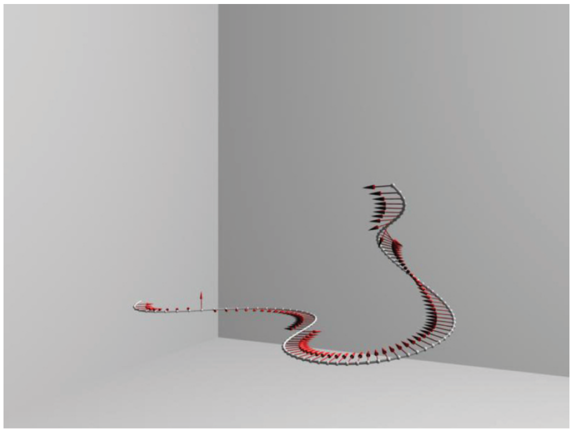


Figure 4: A FAMILY OF DUAL STREAM FUNCTION SURFACES. THE INTERSECTION OF THE STREAM SURFACES HERE FORM STREAMLINES.

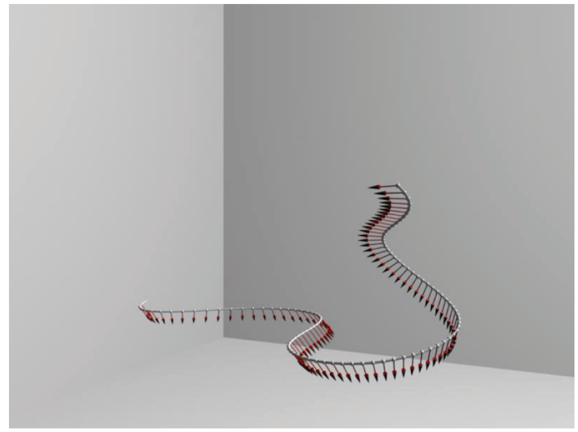
1. For a given start point, find the cell that contains that point.
2. Construct the *fg* diagram for that cell.
3. From the *fg* diagram, find the entry and exit faces for the streamlines.
4. Go the the neighbouring cell and repeat.

The *fg* diagrams can be constructed using one of the four *normalised fg* diagrams shown in Fig.5, depending on the number of inflow, outflow, and no-flow faces in the tetrahedron. Case (a) is used when there are two inflow and two outflow faces; case (b) when there are three inflow and one outflow faces (or vice versa); case (c) when there is one no-flow face, and case (d) when there are two no-flow faces. Finding the inlet and exit faces can be done by computing the barycentric coordinates of the streamline with respect to the nodes of each face, which involves inverting a three by three matrix for each face tested. Only faces known to be outflow faces need be tested. This compares favourably with a numerical integration scheme which requires the inversion of a four by four matrix for each cell the streamline visits. The algorithm terminates when the streamline reaches a boundary or reaches a face already visited. This prevents the streamline circulating forever in a re-circulation zone.

The streamlines are rendered by connecting the inlet and outlet points of each tetrahedron with a straight line. A smoother streamline can be created by then passing an interpolating spline through the points of the streamline. Streamlines computed using this technique can be seen in Fig.6. The flow is through a ventricular assist device [Were and



(a) The Frenet frame of a spine curve. Only normal vectors are shown.



(b) A rotation minimizing frame (RMF) of the same curve in (a). Only reference vectors are shown.

(c) A snake modeled using the RMF in (b).

Fig. 2. An example of using the RMF in shape modeling.

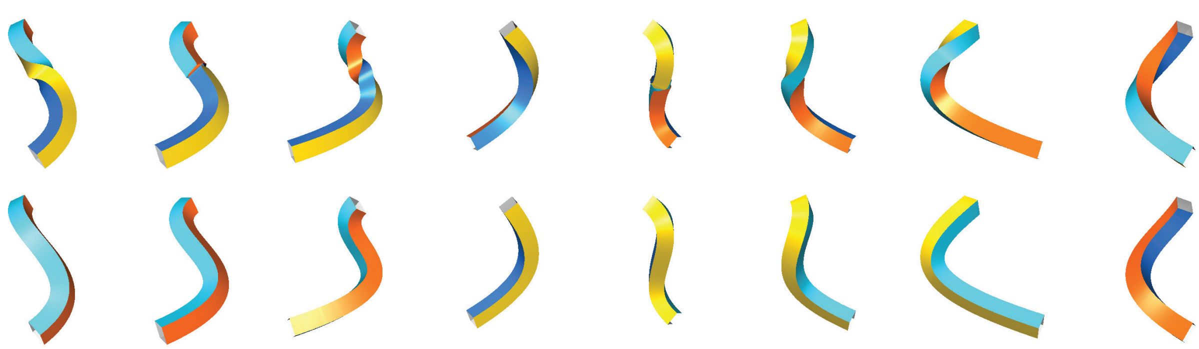


Fig. 3. Sweep surfaces showing moving frames of a deforming curve: the Frenet frames in the first row and the RMF in the second row.

Scope of the presentation

- The form of the stationary Euler equations in the frame of the dual stream function representation

  $\mathbf{u}(x,y,z) = \nabla\lambda(x,y,z) \times \nabla\mu(x,y,z).$

- Lie point symmetries both for Beltrami fields and for force-free fields.

- The equivalence transformation for the complete system.

- Considering λ and μ as local coordinates of a $2D$ Riemannian manifold M^2 we give the classification of M^2 in terms of algebraic surfaces in general locally.

The dual stream function representation

The stationary incompressible Euler equations

$$(\mathbf{u}, \nabla)\mathbf{u} = -\nabla p,$$

$$\operatorname{div} \mathbf{u} = 0, \tag{1}$$

in a $D \subset \mathbb{R}^3$ can be equivalently rewritten in the compact form

$$\mathbf{u} \times \operatorname{curl} \mathbf{u} = \nabla H \tag{2}$$

where $H = p + \|\mathbf{u}\|^2/2$ is the Bernoulli function.

The Beltrami *property* $\mathbf{u} \times \operatorname{curl} \mathbf{u} = \mathbf{0}$ leads to an alignment of the velocity $\mathbf{u}$ and its vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ on all critical H-levels:

$$\operatorname{curl} \mathbf{u} = \kappa \cdot \mathbf{u}, \tag{3}$$

$\kappa : D \to \mathbb{R}$ is a function of the coordinates $\mathbf{x}$ in general. In the lexis of MHD such fields are called *force-free fields*. When κ is a constant, i.e. $\mathbf{u}$ is an eigenfunction of the curl operator then such class of fields are called *Beltrami fields*.

The transition to the dual stream function representation for the velocity field, using the potential variables λ and μ, is defined as (Yih)

$$\mathbf{u} = \nabla\lambda \times \nabla\mu. \tag{4}$$

The family of $\lambda(x, y, z) =$ const. and $\mu(x, y, z) =$ const. stratify space: the flow lines locally coincide with these surface intersections. Using (2) and (4), the Euler equations take the following form

$$(\boldsymbol{\omega}, \nabla\,\mu) = -\frac{\partial H}{\partial \lambda}, \tag{5}$$

$$(\boldsymbol{\omega}, \nabla\,\lambda) = \frac{\partial H}{\partial \mu}. \tag{6}$$

The dual stream function representation is closely related to the Clebsch-potentials: $\mathbf{u} = \nabla\phi + \alpha\nabla\beta$. The Clebsch representation is only defined locally, and is not unique, but admits a gauge group. These gauge transformations turn out to be canonical transformations (Zaharov, Kuznetsov). These transformations induce a family of gauge manifolds M^2 that preserve the element of area: $\sqrt{\det(g_{ij})}(d\alpha \wedge d\beta) = \sqrt{\det(g'_{ij})}(d\alpha' \wedge d\beta')$, where g_{ij} is a metric tensor of M^2.

First, we expose symmetries of the Euler equations have been calculated by Ovsyannikov in the frame of the modified Clebsch variables representation:

$$\mathbf{u} = \nabla\phi + \frac{1}{2}(\alpha\nabla\beta - \beta\nabla\alpha)\,, \tag{7}$$

where all fields depend on (t, x, y, z).

The equations of motion change to the form:

$$D\alpha = \frac{\partial f}{\partial \beta}, \quad D\beta = -\frac{\partial f}{\partial \alpha}, \quad 2\Delta\phi + \alpha\Delta\beta - \beta\Delta\alpha = 0, \tag{8}$$

$$D = \frac{\partial}{\partial t} + (\mathbf{u}, \nabla), \tag{9}$$

f is a function of the variables (t, α, β):

$$p = -\phi_t - 1/2(\alpha\beta_t - \beta\alpha_t) - 1/2|\mathbf{u}|^2 - f$$

Clebsch identity: $\nabla f = (D\alpha)\nabla\beta - (D\beta)\nabla\alpha$. In the case of $f \equiv 0$

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

$$X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial \alpha} + \beta\frac{\partial}{\partial \beta} + \phi\frac{\partial}{\partial \phi}, \quad X_4 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},$$

$$X_5 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad X_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad X_7 = b_1(t)\frac{\partial}{\partial x} + b_1'(t)x\frac{\partial}{\partial \phi},$$

$$X_8 = b_2(t)\frac{\partial}{\partial y} + b_2'(t)y\frac{\partial}{\partial \phi}, \quad X_9 = b_3(t)\frac{\partial}{\partial z} + b_3'(t)z\frac{\partial}{\partial \phi},$$

$$X_{10} = F_\beta\frac{\partial}{\partial \alpha} - F_\alpha\frac{\partial}{\partial \beta} + \frac{1}{2}\left(\alpha F_\alpha - \beta F_\beta - 2F\right)\frac{\partial}{\partial \phi}, \quad X_{11} = h(t)\frac{\partial}{\partial \phi},$$

$b_i(t)$, $h(t)$ and $F(\alpha, \beta)$ are arbitrary functions. This basis forms an infinite-dimensional Lie algebra which contains the maximal finite-dimensional Lie subalgebra with dim = 23. X_3 and X_{10} transform the variables α and β. Operator X_{10} generates the following transformation:

$$\frac{d\alpha'}{da} = F_{\beta'}, \quad \frac{d\beta'}{da} = -F_{\alpha'}, \quad \frac{d\phi'}{da} = \frac{1}{2}\left(\alpha' F_{\alpha'} + \beta' F_{\beta'} - 2F\right), \quad t' = t,$$

$$\alpha'(0) = \alpha, \quad \beta'(0) = \beta, \quad \phi'(0) = \phi.$$

The system of equations for the Clebsch-pair (α, β)

$$\frac{d\alpha'}{da} = F_{\beta'}, \quad \frac{d\beta'}{da} = -F_{\alpha'}, \tag{10}$$

forms a Hamiltonian system, $F(t, \alpha, \beta)$ is the first integral and

$F(t, \alpha'(a), \beta'(a))$ is independent of the group parameter a:

$F(t, \alpha', \beta') = F(t, \alpha, \beta)$, and therefore system (10) is integrable. We

have a transformation

$$\alpha' = \alpha'(t, \alpha, \beta, a), \quad \beta' = \beta'(t, \alpha, \beta, a), \quad \phi' = \phi + \phi'(t, \alpha, \beta, a),$$

which preserves the element of area on a two-dimensional

Riemannian manifold M^2.

Symmetries of the stationary Euler equations in the frame of dual stream function

First, we look at the Lie-point symmetries both for the Beltrami and force-free fields.

$$X^B = \xi_x \frac{\partial}{\partial x} + \xi_y \frac{\partial}{\partial y} + \xi_z \frac{\partial}{\partial z} + \eta_\lambda \frac{\partial}{\partial \lambda} + \eta_\mu \frac{\partial}{\partial \mu}, \tag{11}$$

the coordinates of X^B depend on the variables (x, y, z, λ, μ).

$$\xi_x = c_4 x + c_1 y + c_2 z + c_6, \quad \xi_y = -c_1 x + c_4 y + c_3 z + c_7,$$

$$\xi_z = -c_2 x - c_3 y + c_4 z + c_5, \quad \eta_\lambda = F_8(\lambda, \mu), \quad \eta_\mu = F_9(\lambda, \mu), \tag{12}$$

under the constraint

$$\frac{\partial}{\partial \lambda} F_8(\lambda, \mu) + \frac{\partial}{\partial \mu} F_9(\lambda, \mu) = c_0, \tag{13}$$

c_i are arbitrary constants, $F_j(\lambda, \mu)$ arbitrary functions. The system (5), (6) also admits the reflection (discrete) symmetries: $(\lambda, \mu) \to (-\lambda, \mu)$ and $(\lambda, \mu) \to (\lambda, -\mu)$. In the set of transformations (12) only the operator

$$X_\infty^B = F_8(\lambda,\mu)\frac{\partial}{\partial\lambda} + F_9(\lambda,\mu)\frac{\partial}{\partial\mu}, \quad \frac{\partial}{\partial\lambda}F_8(\lambda,\mu) = -\frac{\partial}{\partial\mu}F_9(\lambda,\mu), \qquad (14)$$

transforms λ and μ. Setting $c_0 = 0$ and introducing a stream function $F(\lambda,\mu)$ as

$$F_8(\lambda,\mu) = \frac{\partial F}{\partial\mu}, \quad F_9(\lambda,\mu) = -\frac{\partial F}{\partial\lambda},$$

we obtain a Hamiltonian system for the transformed variables (λ',μ').

For the general equivalence transformation of the system (5), (6)

the corresponding infinitesimal operator is of the form

$$X^E = \xi_x \frac{\partial}{\partial x} + \xi_y \frac{\partial}{\partial y} + \xi_z \frac{\partial}{\partial z} + \eta_\lambda \frac{\partial}{\partial \lambda} + \eta_\mu \frac{\partial}{\partial \mu} + \eta_H \frac{\partial}{\partial H}, \qquad (15)$$

the coordinates of X^E are functions of $(x, y, z, \lambda, \mu, H)$.

$$\xi_x = k_2 x + k_3 y + k_5 z + k_6, \quad \xi_y = -k_3 x + k_2 y - k_1 z + k_7,$$

$$\xi_z = -k_5 x + k_1 y + k_2 z + k_4, \quad \eta_\lambda = F_8(\lambda, \mu), \quad \eta_\mu = F_9(\lambda, \mu), \quad \eta_H = k_0 H$$

under the constraint

$$2\frac{\partial}{\partial\lambda}F_8(\lambda,\mu)+2\frac{\partial}{\partial\mu}F_9(\lambda,\mu)=4k_2+k_0, \tag{16}$$

k_i are arbitrary constants. The relevant operator which transforms the variables λ and μ is again of the form (14), i.e. the equivalence transformation obtained generates a gauge group of transformations for λ, μ.

Therefore the transformations of λ and μ are defined by

$$\frac{\mathrm{d}\lambda'}{\mathrm{d}a} = F_{\mu'}, \quad \frac{\mathrm{d}\mu'}{\mathrm{d}a} = -F_{\lambda'}, \quad \lambda'(0) = \lambda, \quad \mu'(0) = \mu. \tag{17}$$

This system is of Hamiltonian type i.e. the transformation for (λ, μ)

$$T^a_{\Phi,\Psi} : \lambda' = \Phi(\lambda, \mu; a), \quad \mu' = \Psi(\lambda, \mu; a), \tag{18}$$

is a canonical. When F is of quadratic form, this transformations coincide with $Sp(2) \simeq SL_2(\mathbb{R})$.

Local structure of M^2

The manifold M^2 in each chart can be equipped by a metric in the conformal form

$$ds^2 = \Lambda^2(\lambda, \mu)(d\lambda^2 + d\mu^2). \tag{19}$$

Let $\mathbf{r} = \mathbf{r}(\lambda, \mu)$ be a local realization of M^2. Consider again the infinitesimal transformation (18) of the variables λ and μ

$$\lambda' = \lambda + \delta\lambda, \quad \mu' = \mu + \delta\mu. \tag{20}$$

We can write

$$\mathbf{r}'(\lambda, \mu) = \mathbf{r}(\lambda, \mu) + \mathbf{R}(\lambda, \mu), \tag{21}$$

$\mathbf{R}(\lambda, \mu)$ denotes an infinitesimal deformation of M^2. Direct calculations for a quadratic form of the Hamiltonian F_U show that M^2 admits sliding along itself. The metric ds^2 is a round.

This result can now be used to classify the shape of M^2 at least locally. The tangent components of the displacement vector $\mathbf{R}(\lambda, \mu)$ we denote by m_α, $\alpha = 1, 2$ belonging to the decomposition $\mathbf{R} = m_\alpha \boldsymbol{\tau}^\alpha$, $\boldsymbol{\tau}^1 = (1, 0)$, $\boldsymbol{\tau}^2 = (0, 1)$. The kinematic system of equations for the field of displacements in the case of sliding the manifold M^2 is

$$\nabla_\alpha m_\beta + \nabla_\beta m_\alpha = 0, \quad \alpha, \beta = 1, 2, \tag{22}$$

$\nabla_\alpha m_\beta = \partial m_\beta / \partial x^\alpha - \Gamma^\lambda_{\alpha\beta} u_\lambda$. We consider the complex function of displacement $\widehat{w} = m_1 + i m_2$ and define for the positive Gaussian curvature $K > 0$ the function

$$W = \frac{\widehat{w}(\widehat{\zeta})}{\sqrt{aK^{1/2}}}, a = a_{11}a_{22} - a_{12}^2 > 0, a_{\alpha\beta} = \mathbf{r}_\alpha \mathbf{r}_\beta, \mathbf{r}_1 = \frac{\partial \mathbf{r}}{\partial \lambda}, \mathbf{r}_2 = \frac{\partial \mathbf{r}}{\partial \mu}, \widehat{\zeta} = \lambda + \mu,$$

which satisfies

$$\partial_{\bar{\zeta}} W + B\overline{W} = 0. \tag{23}$$

Here

$$B = \frac{1}{4}\left(\Gamma_{22}^1 - \Gamma_{11}^1 + 2\Gamma_{12}^2\right) - \frac{i}{4}\left(\Gamma_{11}^2 - \Gamma_{22}^2 + 2\Gamma_{12}^1\right). \tag{24}$$

The Christoffel symbols equal

$$\frac{1}{4}\left(\Gamma^1_{22} - \Gamma^1_{11} + 2\Gamma^2_{12}\right) = 0, \quad \frac{i}{4}\left(\Gamma^2_{11} - \Gamma^2_{22} + 2\Gamma^1_{12}\right) = 0,$$

since the metric ds^2 is round. Therefore W is a holomorphic function

and the function of displacement $\widehat{w} = \widehat{w}(\widehat{\zeta})$ reads

$$\widehat{w}(\widehat{\zeta}) = \sqrt{aK^{1/2}}\Theta(\overline{\zeta}),$$

$\Theta(\widehat{\zeta})$ is a holomorphic function. Vekua: *The condition $B \equiv 0$ is satisfied for the second-order algebraic surfaces of a positive Gaussian curvature and only for such surfaces.*

We can claim that in each chart where M^2 has a positive Gaussian curvature the surface M^2 takes locally the form of an ellipsoid (in particular a sphere), bicameral hyperboloid or paraboloid. These surfaces are invariant under the discrete symmetries $(\lambda, \mu) \to (-\lambda, \mu)$ and $(\lambda, \mu) \to (\lambda, -\mu)$ admitted by the Beltrami fields. Notice that for $K < 0$ there is no similar result to classify surfaces of negative Gaussian curvature.

Distance in the frame of (λ, μ)-variables

Consider the set $\Omega(M^2, P, Q)$ of piecewise smooth curves $\gamma : I \to M^2$

with fixed endpoints $\gamma(0) = P$ and $\gamma(1) = Q$. Let

$L_\gamma : \Omega(M^2, P, Q) \to \mathbb{R}$ be the functional of length, then

$$d(P,Q) = \min_{\gamma \in \Omega(M^2; P,Q)} L_\gamma, \quad L_\gamma = \int_\gamma \sqrt{\Lambda^2(\rho)(\lambda_\tau^2 + \mu_\tau^2)} d\tau,$$

defines $d : M^2 \times M^2 \to \mathbb{R}$ where τ denotes the natural parameter of γ.

Simplest conservation laws follows from the Noether theorem.

Instead of (λ, μ), we consider (ρ, ϕ): $\lambda = \exp \rho \cdot \cos \phi$, $\mu = \exp \rho \cdot \sin \phi$.

The metric ds^2 on M^2 then reads:

$$ds^2 = \mathcal{F}(\rho)(d\rho^2 + d^2\phi). \tag{25}$$

The isometric transformation $g_a : (\rho, \phi) \mapsto (\rho, \phi + ca)$ is admitted.

The infinitesimal operator (Killing field) is $X = c\frac{\partial}{\partial \phi}$. This operator

generates a local Hamiltonian flow with a linear Hamiltonian.

We show how invariance of the function $d(P,Q)$ is realized when the transformation $T^a_{\Phi,\Psi}$ is linear, i.e. when it coincides with the symplectic group $Sp(2)$.

Consider again the functional L_γ. We can account that

$$\Lambda^2(\rho)(\lambda_\tau^2 + \mu_\tau^2) = 1 \quad \text{along the curve } \gamma. \tag{26}$$

Therefore

$$L_\gamma = \int_\gamma 1 \cdot d\tau = \tau_\gamma,$$

τ_γ denotes the length of the curve γ. Instead of the vector (λ, μ), we consider the co-vector (τ_λ, τ_μ)

$$\lambda_\tau = \frac{\tau_\lambda}{\Lambda^2}, \quad \mu_\tau = \frac{\tau_\mu}{\Lambda^2}.$$

Then (26) simplifies to

$$\tau_\lambda^2 + \tau_\mu^2 = \Lambda^2(\rho). \tag{27}$$

This equation is of eikonal-type. Therefore in order to find symmetries of the functional L_γ we can consider symmetries admitted by equation (27) that leaves τ invariant.

An equivalence transformation admitted by (27) is a point

transformation (λ, μ, u^1, u^2)-space, $u^1 = \tau$, $u^2 = \Lambda^2$. Infinitesimally, we

look for an operator in the following form (Megrabov, Meleshko)

$$Y = \xi(\lambda, \mu, u^1, u^2)\frac{\partial}{\partial \lambda} + \eta(\lambda, \mu, u^1, u^2)\frac{\partial}{\partial \mu} + \Xi^i(\lambda, \mu, u^1, u^2)\frac{\partial}{\partial u^i} \qquad (28)$$

The coefficients of the operator Y are

$$Y = f(\lambda, \mu)\frac{\partial}{\partial \lambda} + g(\lambda, \mu)\frac{\partial}{\partial \mu} + h(\tau)\frac{\partial}{\partial \tau} + 2\left(\frac{dh}{d\tau} - f_\lambda(\lambda, \mu)\right)\Lambda^2\frac{\partial}{\partial \Lambda^2}.$$

The Lie subalgebra of Y which admits the scalar invariant τ is

$$X = f(\lambda,\mu)\frac{\partial}{\partial\lambda} + g(\lambda,\mu)\frac{\partial}{\partial\mu} - 2f_\lambda(\lambda,\mu)\Lambda^2\frac{\partial}{\partial\Lambda^2}. \tag{29}$$

$f(\lambda,\mu)$, $g(\lambda,\mu)$ satisfy the Cauchy-Riemann differential equations $f_\lambda = g_\mu$, $f_\mu = -g_\lambda$. To analyse the fine structure of this equivalence transformation generated by X, we switch from (λ,μ) to complex variables $z = \lambda + i\mu$ and $\bar{z} = \lambda - i\mu$.

$$X = \psi(z)\frac{d}{dz} + \bar{\psi}(\bar{z})\frac{d}{d\bar{z}} - \psi_z(z)\Lambda^2(z\bar{z})\frac{d}{d\Lambda^2} - \bar{\psi}_{\bar{z}}(\bar{z})\Lambda^2(z\bar{z})\frac{d}{d\Lambda^2}, \tag{30}$$

$\psi = f + ig$, $\bar{\psi} = f - ig$, the holomorphic function ψ only shows the dependence $\psi(z,\bar{z}) \equiv \psi(z)$, $\bar{\psi}(z,\bar{z}) \equiv \bar{\psi}(\bar{z})$. The basis of X (30) is

$$k_n = -z^{n+1}\frac{d}{dz} - (n+1)z^n \Lambda^2 \frac{d}{d\Lambda^2}, \quad \bar{k}_n = -\bar{z}^{n+1}\frac{d}{d\bar{z}} - (n+1)\bar{z}^n \Lambda^2 \frac{d}{d\Lambda^2}, \quad n \in \mathbb{Z}.$$

The factor Λ^2 is transformed to

$$\Lambda^{2^*} = \frac{\Lambda^2}{z'_z}, \text{ and } \Lambda^{2'} = \frac{\Lambda^{2^*}}{\bar{z}'_{\bar{z}}} = \frac{\Lambda^2}{z'_z \bar{z}'_{\bar{z}}}, \tag{31}$$

The linear hull of $k_n \oplus \bar{k}_n$ over $\mathbb{C}$ we denote by $\mathbf{W}$.

$$[k_n, k_m] = (n-m)k_{n+m}, \quad [\bar{k}_n, \bar{k}_m] = (n-m)\bar{k}_{n+m}, \quad [k_n, \bar{k}_m] = 0.$$

The group of $z \mapsto z'$ consists in the set Mb of Möbius transformations φ:

$$\varphi(z) = \frac{az+b}{cz+d}, \text{ with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})/\{\pm 1\}, \text{ and } cz+d \neq 0. \quad (32)$$

For the transformations $\bar{z} \mapsto \bar{z}'$ the corresponding conformal group we denote by $\overline{\text{Mb}}$. The set Mb ($\overline{\text{Mb}}$) forms a group in which the group operation coincides with the matrix multiplication in $SL_2(\mathbb{C})$.

$SL_2(\mathbb{C})$ coincides with the symplectic group $Sp(2)$ which is generated by the Hamiltonian flow with a quadratic Hamiltonian function F_U. The construction of the conformal group makes it possible to implement the notions of current and charge as they arise in conformal field theory. In order to see this, we consider the action functional of the curve γ

$$E_\gamma = \int_\gamma \Lambda^2(\rho)(\lambda_\tau^2 + \mu_\tau^2)d\tau,$$

rewritten in the form

$$E_\gamma = \int_\gamma \Lambda^{-2}(\rho)(\tau_\lambda^2 + \tau_\mu^2)d\tau, \quad \tau_\lambda^2 + \tau_\mu^2 = \Lambda^2(\rho). \tag{33}$$

E_γ admits the same Lie algebra. The classical energy-momentum tensor is defined by

$$T_{ik} = g_{kl} f^\alpha_{x^i} \frac{\partial L}{\partial f^\alpha_{x^l}} - g_{ik} L, \tag{34}$$

For the functional (33) with the scalar field τ this tensor takes the form

$$T_{11} = 2\tau_\lambda^2 - \Lambda^2 \cdot 1, \quad T_{22} = 2\tau_\mu^2 - \Lambda^2 \cdot 1, \quad T_{12} = T_{21} = 2\tau_\lambda \tau_\mu. \quad (35)$$

T_{ik} is a traceless tensor: $T_{11} \pm T_{22} = 2(\tau_\mu^2 + \tau_\lambda^2 - \Lambda^2) = 0$. Using $\boldsymbol{\nu} = (\lambda, \mu)$, $\delta\boldsymbol{\nu} = (\delta\lambda, \delta\mu)$ the current is defined by $j_\sigma = T_{\sigma\kappa}\delta\nu^\kappa$ and has an vanishing divergence due to the tracelessness of $T_{\sigma\kappa}$. Considering the infinite dimensional Lie pseudo-group $\mathbf{G}$ generated by $\mathbf{W}$, we have an infinite number of conserved currents j_σ. For the conformal group, however, we only have a finite number of conserved currents j_σ.

In the complex coordinates, the conserved currents $j_\sigma = T_{\sigma\kappa}\delta\nu^\kappa$ are transformed to $j_z = T_{zz}\epsilon(z)$ and $j_{\bar{z}} = T_{\bar{z}\bar{z}}\bar{\epsilon}(\bar{z})$. Expanding $\epsilon(z)$, $\bar{\epsilon}(\bar{z})$ into

$$\epsilon(z) = \sum_n \epsilon_n z^{n+1}, \quad \bar{\epsilon}(\bar{z}) = \sum_n \bar{\epsilon}_n \bar{z}^{n+1},$$

gives an infinite number of conserved currents $j_z^n = T_{zz}\epsilon_n z^{n+1}$, $j_{\bar{z}}^n = T_{\bar{z}\bar{z}}\bar{\epsilon}_n \bar{z}^{n+1}$ and a finite number of these for $Sp(2)$, by fixing $n = 0, \pm 1$.

Since $T(z)$, $\bar{T}(\bar{z})$ are holomorphic functions

$$T(z) = \sum_n L_n z^{-n-2}, \quad \bar{T}(\bar{z}) = \sum_n \bar{L}_n \bar{z}^{-n-2}, \tag{36}$$

can be inverted

$$L_n = \frac{1}{2\pi i} \int T(z) z^{n+1} dz, \quad \bar{L}_n = \frac{1}{2\pi i} \int \bar{T}(\bar{z}) \bar{z}^{n+1} d\bar{z},$$

the exponent $-n-2$ in (36) was chosen such that for the scale transformation $z \mapsto \lambda^{-1} z$, under which $T(z) \mapsto \lambda^2 T(\lambda^{-1} z)$, we get $L_{-n} \mapsto \lambda^n L_{-n}$, and $\bar{L}_{-n} \mapsto \bar{\lambda}^n \bar{L}_{-n}$. Conformal charges Q_ϵ, $Q_{\bar{\epsilon}}$ are then determined by

$$Q_\epsilon = \sum_n \epsilon_n L_n, \quad Q_{\bar{\epsilon}} = \sum_n \bar{\epsilon}_n \bar{L}_n,$$

which present an infinite number of conserved quantities for the Lie pseudo-group $\mathbf{G}$ due to the independence of L_n and $\bar{L}_n$ on a loop of integration, and correspondingly a finite number of charges for the symplectic group $Sp(2)$.

We found out that τ (or the distance $d(P, Q)$) is a scalar invariant of the group $Sp(2)$.