Collapse in hydrodynamics and the Kolmogorov spectrum

E.A. Kuznetsov

Lebedev Physical Institute, Moscow, Russia

OUTLINE

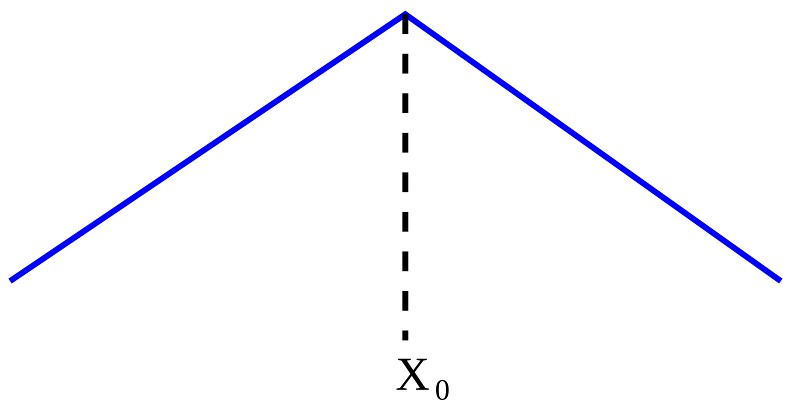
- Motivation: collapses and turbulent spectra
- Relabeling symmetry and Cauchy invariant
- Vortex line representation and new Cauchy invariant
- Breaking of vortex lines
- Numerical experiment

Collapses and turbulent spectra

It is well known that singularities give the power type behavior of the Fourier amplitudes that provides appearance of power tails for turbulent spectra.

Examples:

- (1958) Phillips spectrum for gravity waves on the fluid surface. Surface singularities are wedges:



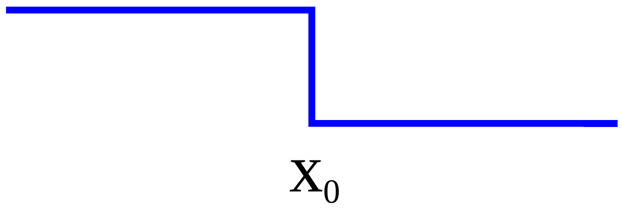
$$\Leftarrow \quad z = \eta(x,t) \Rightarrow$$

$$\eta_{xx} \sim \delta(x - x_0)$$

or $\eta_k \sim k^{-2}$. Hence one can get the Phillips spectrum:

$$\varepsilon_\omega \sim \omega^{-5},\ \omega = \sqrt{gk}.$$

- (1973) Kadomtsev-Petviashvili spectrum. According to KP acoustic turbulence is a random set of shocks:



$$\rho_x \sim \delta(x - x_0),$$
$$\rho_k \sim k^{-1} \;\Rightarrow$$
$$\varepsilon_\omega \sim \omega^{-2}.$$

- (1941) Kolmogorov spectrum, i.e. the energy distribution of the velocity fluctuations in the inertial interval ($Re \gg 1$),

$$E_k \sim P^{2/3} k^{-5/3}$$

where P is the energy flux. This spectrum can be obtained from the dimensional analysis.

Collapses and turbulent spectra

- Using this analysis one can get that the energy transfer time T from large scales L to dissipative ones is finite and defined by L and P : $T \sim L^{2/3}P^{-1/3}$.
- Distribution of velocity fluctuations

  $v \sim P^{1/3}r^{1/3}$

  Respectively, for vorticity $\Omega = \text{curl}\,\mathbf{v}$ we have:

  $$\Omega \sim P^{1/3}r^{-2/3}.$$

  Thus, for Ω we have singularity at $r \to 0$, besides T is finite.

  Question: Is it a real singularity?

Temporal behavior of vorticity at the collapse point - naive arguments

The Euler equation for vorticity $\mathbf{\Omega}$

$$\frac{d\mathbf{\Omega}}{dt} = (\mathbf{\Omega}\cdot\nabla)\mathbf{v}, \;\; \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{v}\cdot\nabla), \;\; \mathrm{div}\,\mathbf{v} = 0.$$

At the maximal point $\mathbf{r} = \mathbf{r}_{max}$ of vorticity $\nabla\Omega|_{\mathbf{r}_{max}} = 0$. Representing as $\mathbf{\Omega} = \Omega\,\tau$ where $\tau = \mathbf{\Omega}/|\mathbf{\Omega}|$ is the unit vector ($\tau^2 = 1$) and noting that $(\tau\cdot\tau_{\mathbf{t}}) = 0$ one can easy get that the maximal value of the vortcity, Ω_{max}, satisties the equation

$$\frac{d\Omega_{max}}{dt} = \frac{\partial v_\tau}{\partial x_\tau}\,\Omega_{max}.$$

This is the exact equation !.

Temporal behavior of vorticity at the collapse point - naive arguments

If $\frac{\partial v_\tau}{\partial x_\tau} = \alpha \Omega_{max}$ then we have the ODE,

$$\frac{d\Omega_{max}}{dt} = \alpha\, \Omega_{max}^2,$$

with the blow-up solution:

$$\Omega_{max} \sim (t_0 - t)^{-1}.$$

From another side, if one assumes that $\Omega_{max} \approx c(t_0 - t)^{-1}$ then

$$\frac{\partial v_\tau}{\partial x_\tau} = \frac{1}{t_0 - t}$$

that can be considered as a checking rule for exact solution.

Relabeling symmetry and Cauchy invariant

It is well known that the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p, \qquad \text{div } \mathbf{v} = 0,$$

has infinite (continuous) number of integrals of motion. These are the so called Cauchy invariants. They can be obtained from the Kelvin theorem

$$\Gamma = \oint_{C[t]} (\mathbf{v} \cdot d\mathbf{l}) = \text{inv}$$

with the movable together with fluid contour $C[t]$. Passing in this integral to the Lagrangian variables,

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t), \ \frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t), \ \mathbf{r}|_{t=0} = \mathbf{a}$$

we arrive at

$$\Gamma = \oint_{C[a]} \dot{x}_i \cdot \frac{\partial x_i}{\partial a_k}\, da_k \ , \ \text{with fixed } C[a].$$

Relabeling symmetry and Cauchy invariant

Hence we get the Cauchy invariants

$$\mathbf{I} = \mathrm{curl}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \equiv \mathbf{\Omega_0}(\mathbf{a})$$

which are constraints in Euler. Conservation of the Cauchy invariants is a consequence of the relabeling symmetry (R. Salmon, 1982). They characterize the frozenness of the vorticity into fluid. The latter means that fluid (Lagrangian) particles can not leave its own vortex line where they were initially.

Relabeling symmetry and Cauchy invariant

This follows from the comparison of the equation for $\delta\mathbf{r}(\mathbf{a},t)$,

$$\frac{d\delta\mathbf{r}}{dt} = (\delta\mathbf{r}\cdot\nabla)\mathbf{v} \equiv \frac{1}{2}[\mathbf{\Omega}\times\delta\mathbf{r}] + \hat{S}\delta\mathbf{r}$$

and

$$\frac{d\Omega}{dt} = \hat{S}\mathbf{\Omega},$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

is the stress tensor, which is responsible for stretching.

Relabeling symmetry and Cauchy invariant

Hence one can see, that if initially $\delta \mathbf{r} \| \mathbf{\Omega}$ then this alignment will remain for all times. However, for $\delta \mathbf{r} \perp \mathbf{\Omega}$ such $\delta \mathbf{r}$ will undergo rotation with the angle velocity $\mathbf{\Omega}/2$.

Thus, the Lagrangian particles have one independent degree of freedom – motion along vortex line. But such a motion does not change the vorticity:

$$\frac{\partial \mathbf{\Omega}}{\partial t} = \text{curl}\,[\mathbf{v} \times \mathbf{\Omega}].$$

Vortex line representation

Thus, the Helmholtz equation contains only one velocity component normal to the vortex line, $\mathbf{v_n}$. The tangent velocity $\mathbf{v}_\tau$ plays a passive role providing incompressibility. Decomposing,$\mathbf{v} = \mathbf{v_n} + \mathbf{v}_\tau$, in the Euler *incompressible* equations leads to the equation of motion of charged *compressible* fluid moving in an electromagnetic field:

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}],$$

where

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} = \text{curl } \mathbf{A}$$

with $\varphi = p + v_\tau^2/2, \;\; \mathbf{A} = \mathbf{v}_\tau$. Thus, two Maxwell equations are satisfied with the gauge: $\text{div}\, \mathbf{A} = -\text{div}\, \mathbf{v}_n \neq 0.$

Vortex line representation

Now perform transform in a new charged *compressible* hydrodynamics to the Lagrangian description:

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r}, t) \text{ with } \mathbf{r}|_{t=0} = \mathbf{a}.$$

Under this transform the new hydrodynamics become the Hamilton equations:

$$\dot{\mathbf{P}} = -\partial h/\partial \mathbf{r}, \qquad \dot{\mathbf{r}} = \partial h/\partial \mathbf{P},$$

$\mathbf{P} = \mathbf{v}_n + \mathbf{A} \equiv \mathbf{v}$ is the generalized momentum, and the Hamiltonian $h = (\mathbf{P} - \mathbf{A})^2/2 + \varphi \equiv p + \mathbf{v}^2/2$ ($\equiv$ the Bernoulli "invariant").

The Kelvin (Liouville) theorem says that $\Gamma = \oint(\mathbf{P} \cdot d\mathbf{R}) = \text{inv}$.

Transform in Γ to new Lagrangian coordinates leads to a new Cauchy invariant :

$$\mathbf{I} = \text{curl}_a \left(P_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \equiv \mathbf{\Omega_0}(\mathbf{a}).$$

Vortex line representation (VLR)

Hence we get

$$\boldsymbol{\Omega}(\mathbf{r}, t) = \frac{(\boldsymbol{\Omega}_0(\mathbf{a}) \cdot \nabla_a)\mathbf{r}(\mathbf{a}, t)}{J(\mathbf{a}, t)}$$

where $\mathbf{P}$ coincides with $\mathbf{v}$ and $J(\mathbf{a}, t) = \partial(\mathbf{r})/\partial(\mathbf{a})$ is the Jacobian of the mapping $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$ defined from

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r}, t), \ \mathbf{r}|_{t=0} = \mathbf{a}.$$

These equations together with

$$\boldsymbol{\Omega}(\mathbf{r}, t) = \mathrm{curl}_r\, \mathbf{v}(\mathbf{r}, t) \text{ and } \mathrm{div}_r\, \mathbf{v}(\mathbf{r}, t) = 0$$

form the complete system of equations in the vortex line representation (Kuznetsov, Ruban (1998), Kuznetsov (2002, 2006)).

The quantity $n = J^{-1}$ plays the role of vortex line density:

$$n_t + \mathrm{div}_r(n\mathbf{v}_n) = 0, \ \ \mathrm{div}_r \mathbf{v}_n \neq 0.$$

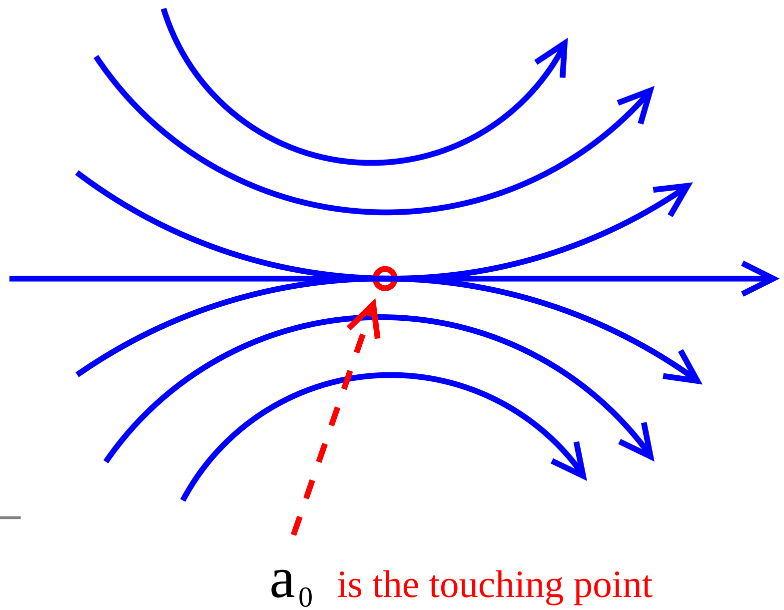
Breaking of vortex lines

REMARK 1: Breaking in gasdynamics corresponds to vanishing J. Because of compressible character of the mapping we can expect breaking of vortex lines.

REMARK 2: Breaking of vortex lines is impossible in 2D and for cylindrically symmetric flows without swirl (Majda, 1990) because $\mathbf{\Omega} \perp \mathbf{v}$ and div $\mathbf{v}_n = 0$, and consequently $J = 1$.

Thus, breaking of vortex line is 3D phenomenon.

Geometrically this results in touching of vortex lines.

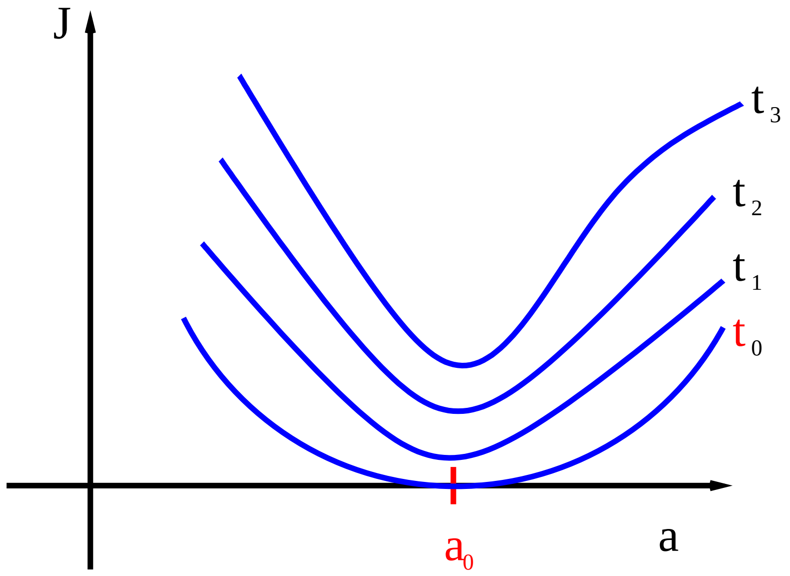


Breaking of vortex lines

Let us assume that breaking takes place. Consider the equation $J(\mathbf{a}, t) = 0$ and find its positive roots $t = \tilde{t}(\mathbf{a}) > 0$. Then the collapse (or touching) time will be

$$t_0 = \min_a \tilde{t}(\mathbf{a}).$$

Near the minimal point $\mathbf{a} = \mathbf{a}_0$ the expansion of J takes



$t_0 > t_1 > t_2 > t_3$

the form:

$$J(a, t) = \alpha(t_0 - t) + \gamma_{ij} \Delta a_i \Delta a_j$$

- concavity condition

$\alpha > 0$,

γ_{ij} is positive definite (nongenerated) matrix,

$\Delta \mathbf{a} = \mathbf{a} - \mathbf{a}_0$.

Breaking of vortex lines

REMARK: The assumption about linear dependence of J_{min} on time t is familiar to the Landau assumption in his theory of the second-order phase transitions.
This expansion results in the self-similar asymptotics for vorticity:

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{(\mathbf{\Omega_0}(\mathbf{a})\cdot\nabla_a)\mathbf{r}|_{a_0}}{\tau(\alpha+\gamma_{ij}\eta_i\eta_j)}, \ \ \eta = \frac{\Delta a}{\tau^{1/2}}, \ \ \tau = t_0 - t.$$

NOTE: This self-similar asymptotics is marginal in the sense of the Beale-Kato-Majda criterion:

$$\int_0^{t_0} \max_r |\mathbf{\Omega}| dt \ \ = \left\{ \begin{array}{c} \infty \text{ for collapsing solutions} \\ < \infty \text{ for noncollapsing solutions} \end{array} \right\}$$

Structure of singularity

1D case

$$v_t + vv_x = 0 \text{ (dust)}, \ p = 0.$$

Solution is given in the implicit form:

$$v = v_0(a), \ x = a + v_0(a)t.$$

Breaking (or gradient catastrophe, or formation of fold) results in the infinite density and velocity gradient ($n = n_0/J$, $\partial v/\partial x = v'_o(a)/J$):

$$J = \frac{\partial x}{\partial a} = \alpha\tau + \gamma a^2 \ \rightarrow x = \alpha\tau a + \frac{1}{3}\gamma a^3.$$

Thus, $a \sim \tau^{1/2}, \ x \sim \tau^{3/2}$!! At $\tau = 0$ we have the singularity:

$$n \sim \frac{\partial v}{\partial x} \sim x^{-2/3}.$$

Structure of singularity

3D case

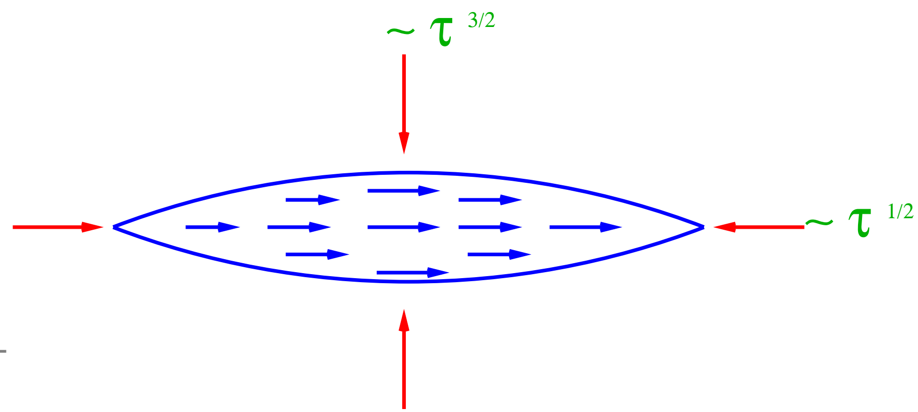
The Jacobian $J = \lambda_1\lambda_2\lambda_3 \to 0$ means that one eigenvalue, say, $\lambda_1 \to 0$ and $\lambda_2, \lambda_3 \to$ const as $t \to t_0$ and $a \to a_0$. Hence it follows that near singular point there are two different self similarities:

along "soft" (λ_1) direction $x_1 \sim \tau^{3/2}$ (like in 1D);

along "hard" (λ_2, λ_3) directions $x_{2,3} \sim \tau^{1/2}$,

so that

$$\mathbf{\Omega} = \frac{1}{\tau}\mathbf{g}\left(\frac{x_1}{\tau^{3/2}}, \frac{x_\perp}{\tau^{1/2}}\right).$$



This results in formation

of pancake structure

(compare with Zeldovich)

Structure of singularity

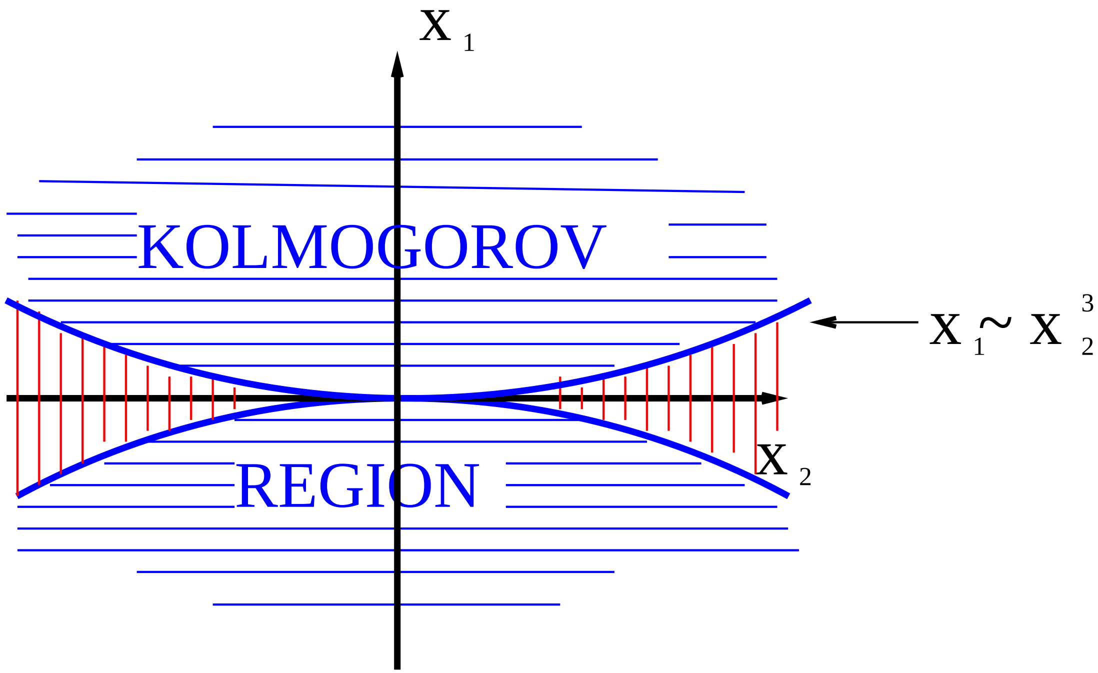
At $\tau = 0$ we get a very anisotropic singularity. The main dependence of $\mathbf{\Omega}$ is connected with x_1-direction:

$$\mathbf{\Omega} \approx \frac{\mathbf{b}}{x_1^{2/3}}$$

with $\mathrm{b} =$ const and KOLMOGOROV index $2/3$!.
This dependence is realized everywhere except regions between two cubic paraboloids $-cx_\perp^3 < x_1 < cx_\perp^3$ In this narrow region vorticity at $\tau = 0$ behaves like

$$\mathbf{\Omega} \approx \frac{\mathbf{b_1}}{x_\perp^2}.$$

Structure of singularity



In Kolmogorov region the vorticity can be estimated as

$$\Omega \sim \frac{P^{1/3}}{x_1^{2/3}}$$

where $P \sim \Omega_0^3 L^2,\ L \sim \gamma^{-1/2}$.

Structure of singularity: velocity

Hence one can get not only the spatial-temporal behavior of the vorticity but also can find similarity for all three components of the velocity. Let us introduce the (local) system of coordinates connected with the maximal point of the vorticity. Let $\hat{x}$ be directed along $\mathbf{\Omega_{max}}$, $\hat{z}$ direction be perpendicular to the vortex sheet, and respectively $\hat{y}$ parallel to the vortex sheet.

According to the exact equation for Ω_{max},

$$\frac{1}{\Omega_{max}}\frac{d\Omega_{max}}{dt} = \frac{\partial v_x}{\partial x}$$

$$\left.\frac{\partial v_x}{\partial x}\right|_{r=r_{max}} = \frac{1}{t_0 - t}$$

.

Structure of singularity: velocity

Because of $x \propto (t_0 - t)^{1/2}$

$$v_x \propto (t_0 - t)^{-1/2}.$$

At small x this component is the linear function relative to x:

$$v_x \approx (t_0 - t)^{-1} x$$

with saturation on the scale $\sim (t_0 - t)^{1/2}$. This is the tangent velocity which does not influence (directly) on the vorticity.

Structure of singularity: velocity

Appearance of the infinitely large tangent velocity is a sequence of the high vorticity region shrinking in z direction. The corresponding component of the velocity can be defined from the incompressibility condition:

$$\frac{\partial v_z}{\partial z} \sim \frac{v_z}{z} \sim \frac{v_z}{(t_0 - t)^{3/2}} \sim \frac{\partial v_x}{\partial x} = (t_0 - t)^{-1} \text{ or } v_z \sim (t_0 - t)^{1/2}.$$

Just this component of the velocity provides the compression of the high vorticity region into the vortex sheet. The component is small and vanishes as $t \to t_0$.

Structure of singularity: velocity

The y- velocity component yields the vorticity

$$\Omega_{max} \approx \left| \frac{\partial v_y}{\partial z} \right| .$$

Hence we have the estimation

$$v_y \sim (t_0 - t)^{1/2}.$$

Another - z-velocity component gives small contribution into Ω_{max}, of order 1.

Numerics

In the main numerics (Pumir & Siggia (1992); Kerr (1993); Pelz (1997), Boratav & R.B. Pelz (1994), Grauer, Marliani, & Germaschewski (1998)) the vorticity maximum blows-up like $(t_0 - t)^{-1}$. Concerning the spatial structure, only a qualitative agreement takes place. The numerics (Brachet, Meneguzzi, Vincent, Politano, Sulem (1992)) showed for the initial conditions (the Taylor-Green vortex and random) the formation of thin vortex layers with high vorticity.
Before Rainer & Sideris (1991) also observed formation of pancake structures for axisymmetric flows with swirl.

Numerics

Recently Orlandi and Co. (P. Orlandi, S. Pirozzoli and G. F. Carnevale, J. Fluid Mech. (2012), vol. 690, pp. 288-320) investigated this problem for initial condition with two counter propagating Lamb vortex dipoles with $\Delta\theta_0 = \pi/2$

$$\Omega_z = -U\kappa \frac{J_1(\kappa r)}{J_o(\kappa a)} \sin(\theta - \theta_0) \text{ if } r \leq a,$$

$$\Omega_z = 0 \text{ if } r > a,$$

where U is the velocity of the dipole, $\kappa a = 3,8317$ is the first positive zero of the Bessel function J_1.

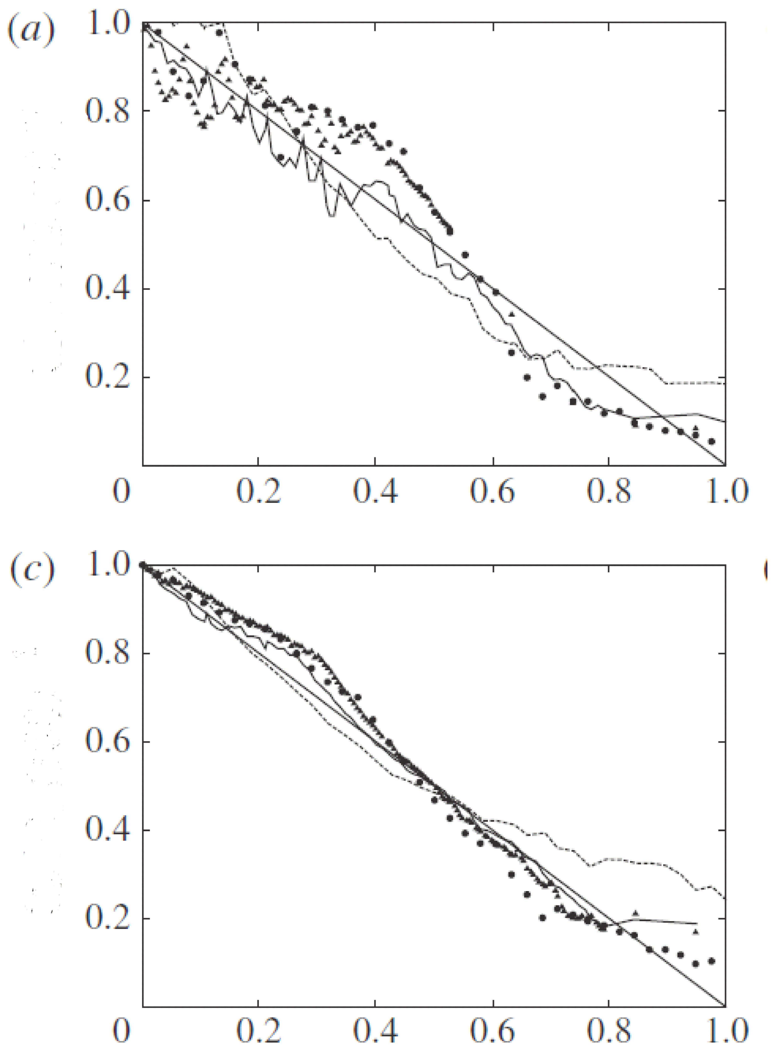
Numerics

These numerical experiments have the high enough spatial resolution: (1024^3 and 1536^3).
These authors found that

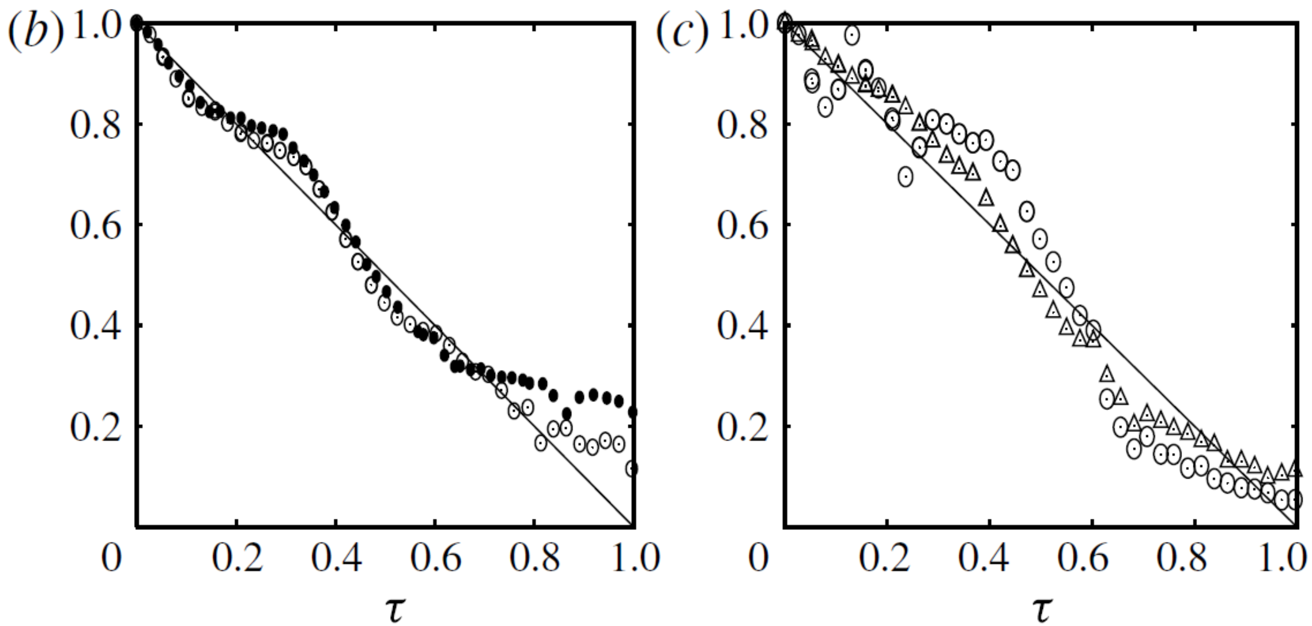
- The vorticity tends to infinilty like $(t_0 - t)^{-1}$,
- The size of collapsing area vanishes as $(t_0 - t)^{1/2}$,
- The velocity component parallel to the vorticity blows-up like $(t_0 - t)^{-1/2}$.
- The region of maximal vorticity represents the vortex sheet.

Numerics

τ-dependences of $(\max|\Omega|)^{-1}$ (a) and $(\max|\Omega_2|)^{-1}$ (c)

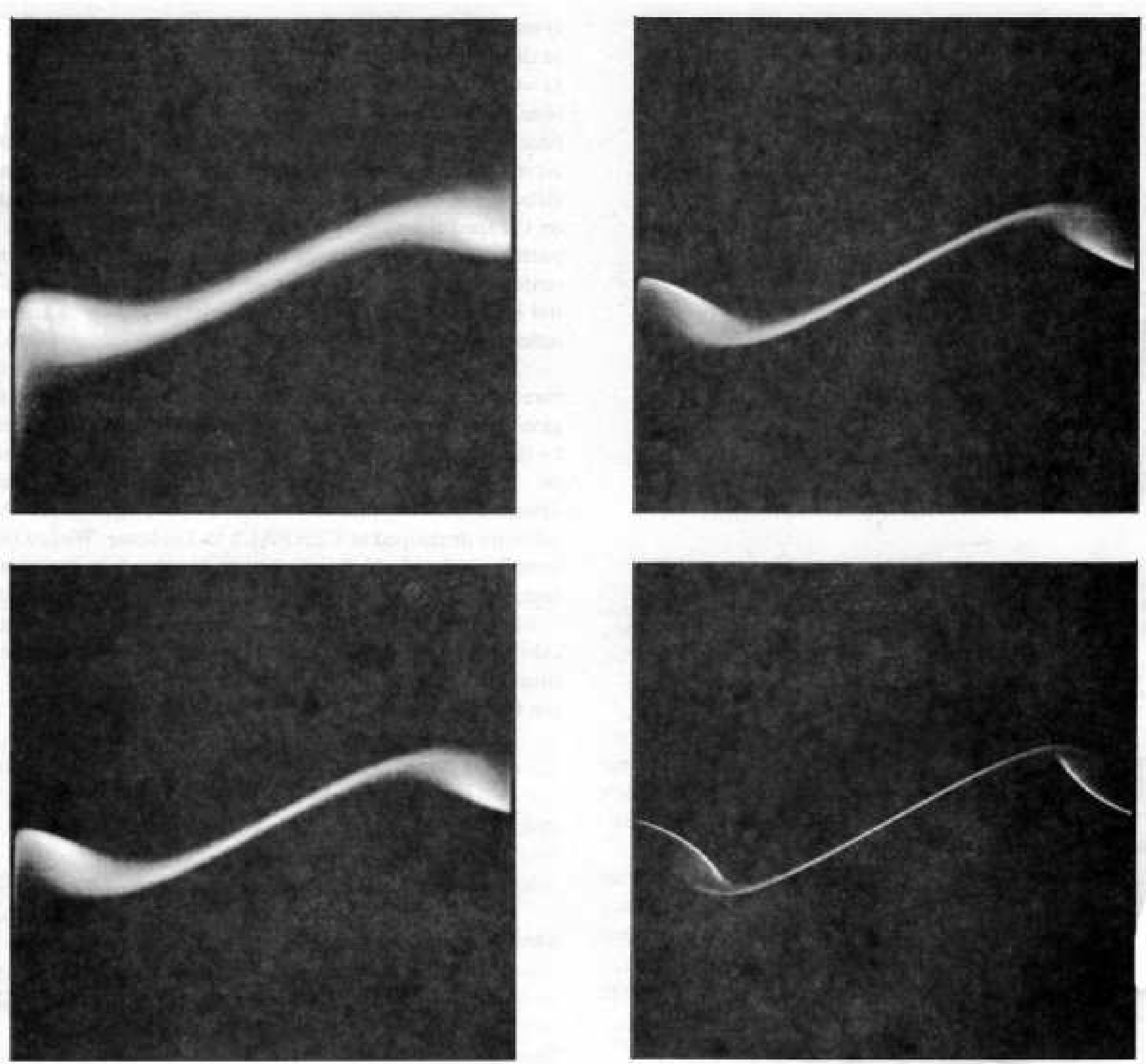


Numerics



Dependences of $(\max|v_2|^{-2})$ (b) and $(\max|\Omega_2|)^{-1}$ (c) relative to normalized time τ

Taylor-Green vortex, Brachet, et. al. (864^3)



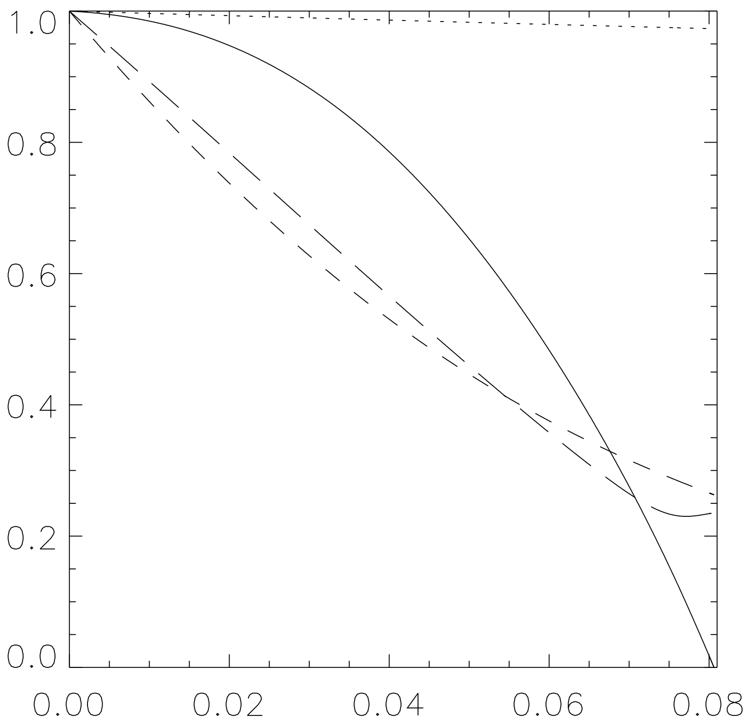
Visualizations in physical space of high vorticity regions for the Taylor–Green vortex at $t = 3$, 3.25, 3.50, and 4.

Numerics

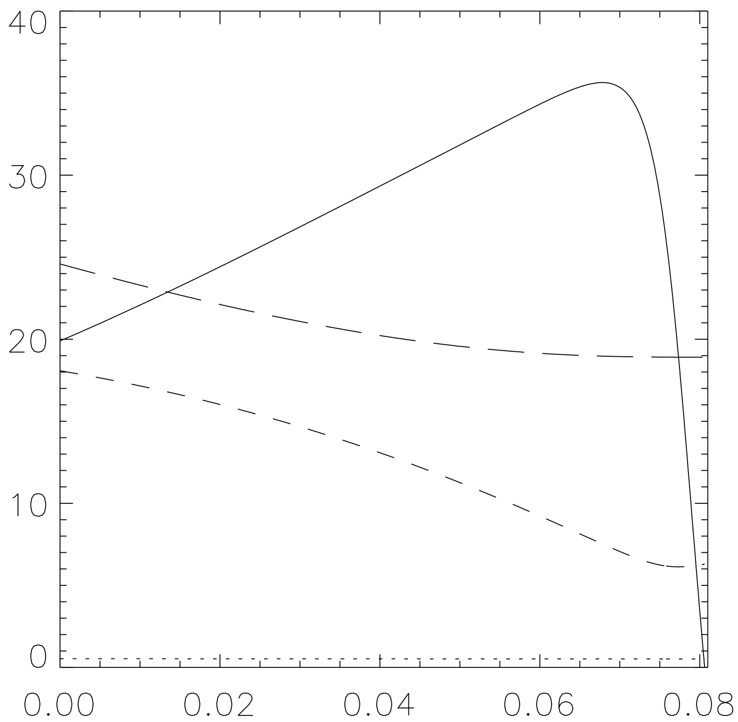
In our numerics (K., Zheligovsky & Podvigina (2001, 2002)) we demostrated that collapse took place due to vanishing of the Jacobian at one separate point (128^3 grids).

"Random" initial conditions

$\min J$

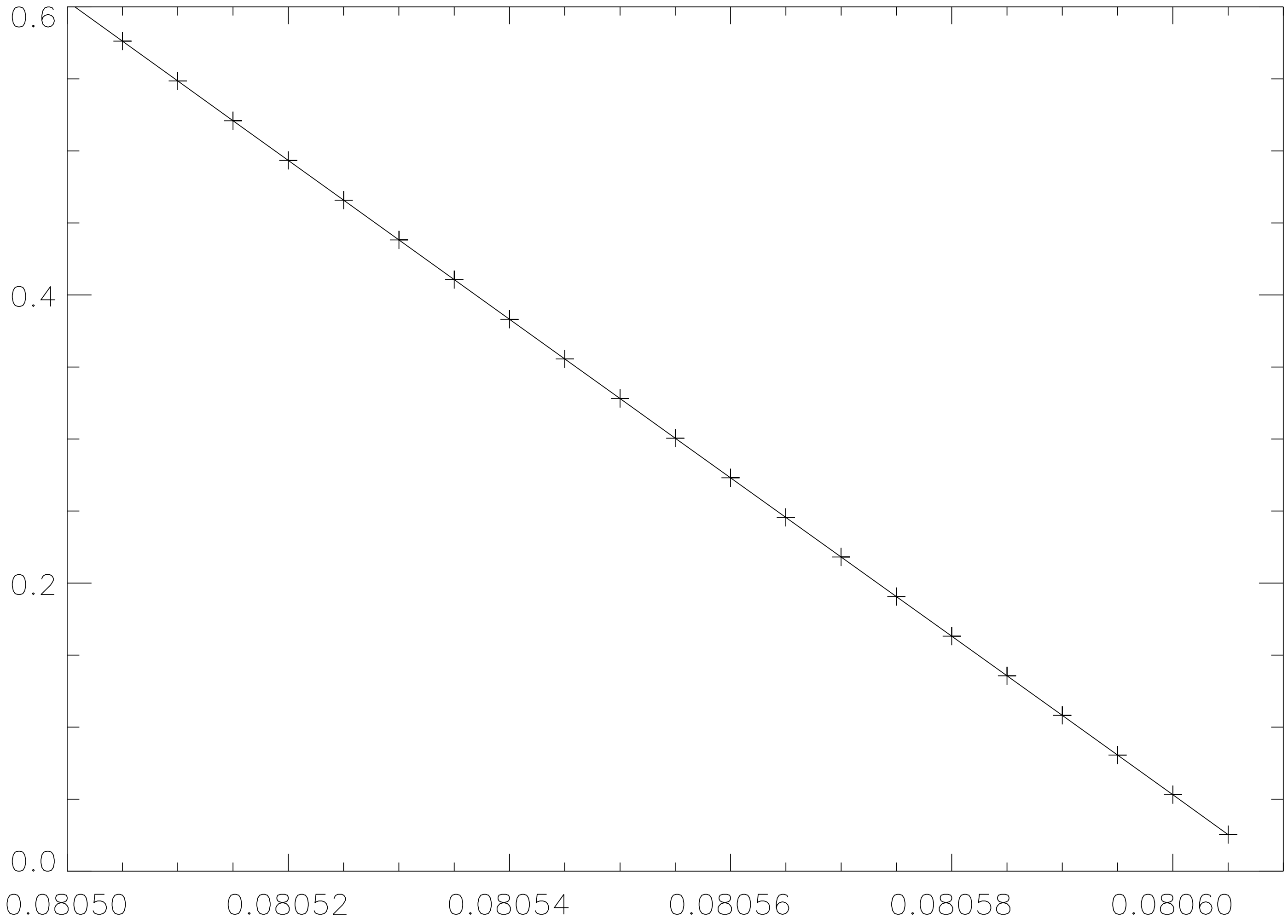


$\min |\Omega|^{-1}$



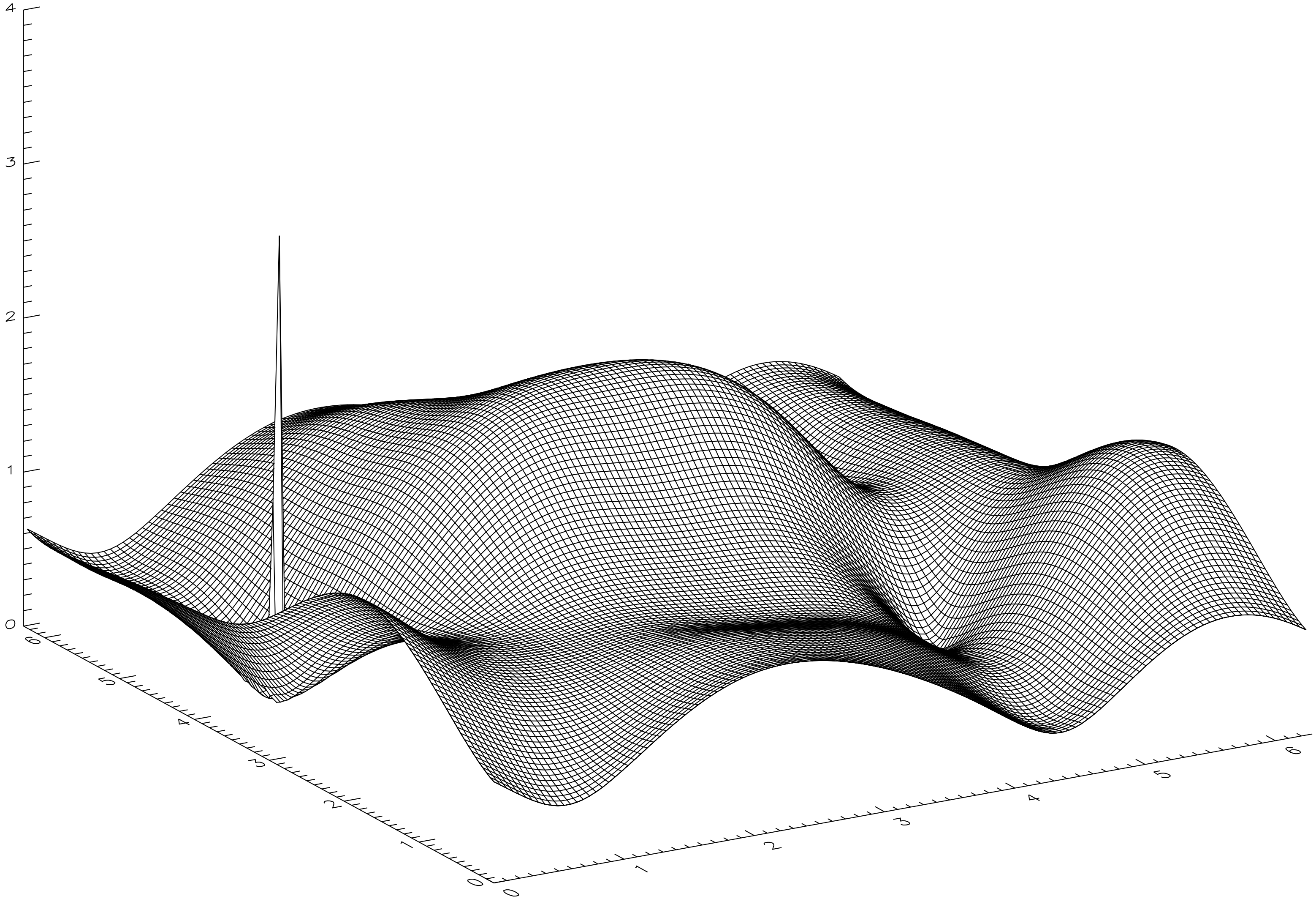
Numerics

$\min J$ (better resolution)



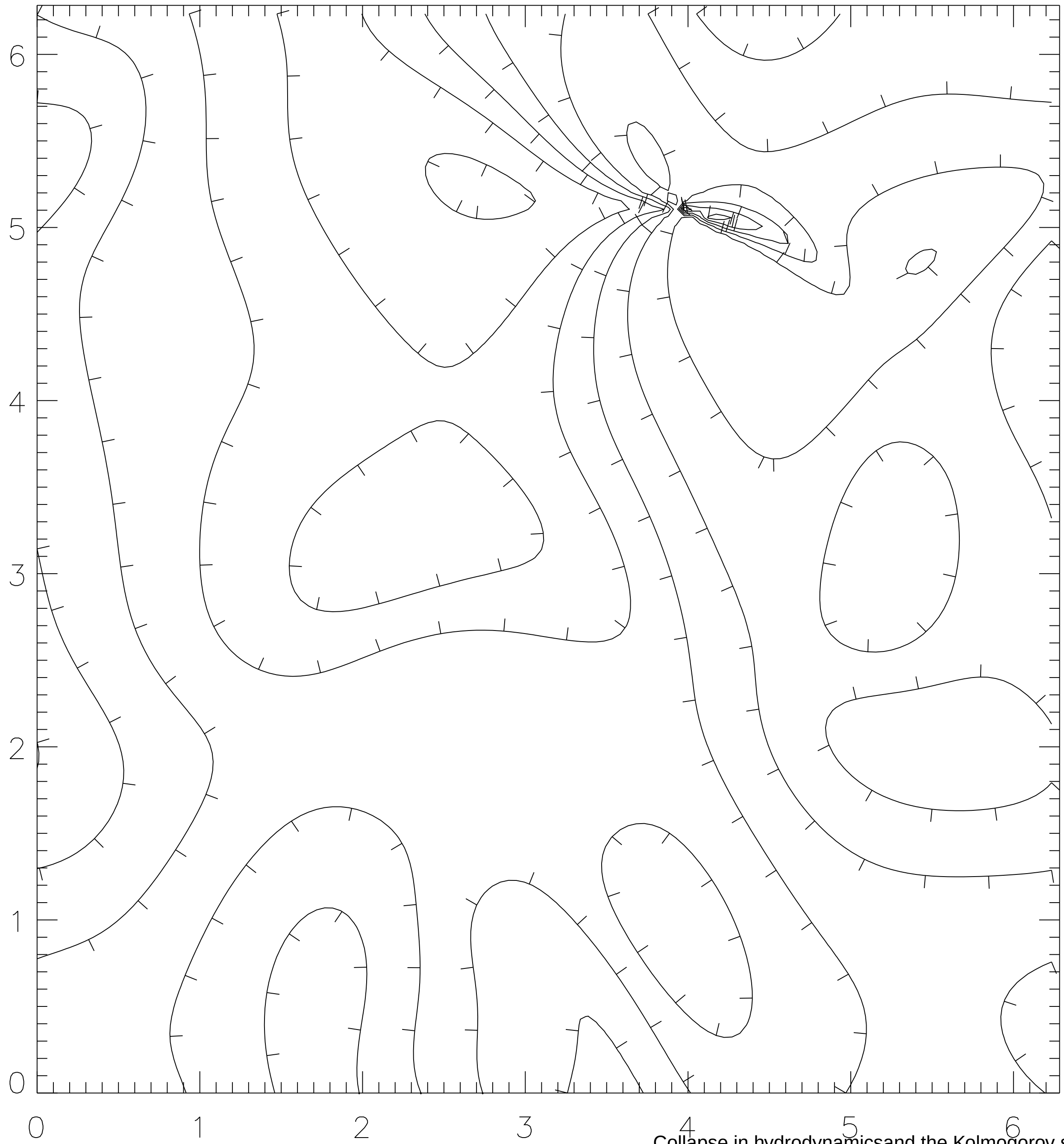
"Random" initial conditions

Dependence of $|\Omega|$ on two coordinates



"Random" initial conditions

Level lines of $|\Omega|$



Numerics

We have checked also that the coefficients γ in the expansion $J(a,t) = \alpha(t_0 - t) + \gamma_{ij}\Delta a_i \Delta a_j$ slightly depend on time as $t \longrightarrow t_0$.

Concluding remarks

- We have presented the arguments both analytical and numerical in favor of existence of collapse in 3D Euler as the process of breaking of vortex lines.
- The Euler equations under VLR transform into the equations for a compressible charge fluid moving in the self-consistent electromagnetic field.
- Compressible character of VLR is the main reason of breaking of vortex lines. For 3D Euler our numerics show that blowup of the vorticity is connected with vanishing of the Jacobian.

Role of viscosity

We have analysed here only the first stage of breaking described by the Euler equations ($Re \gg 1$).
Near singularity we should use the Navier-Stokes (NS) equations.
The VLR application to the NS SPLITS inertial dynamics:

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r},t),\ \mathbf{r}|_{t=0} = \mathbf{a},$$

$$\mathbf{\Omega}(\mathbf{r},t) = \mathrm{curl}_r\, \mathbf{v}(\mathbf{r},t),\ \mathrm{div}_r\, \mathbf{v}(\mathbf{r},t) = 0$$

$$\mathbf{\Omega}(\mathbf{r},t) = \frac{(\mathbf{\Omega}_0(\mathbf{a},t)\cdot\nabla_a)\mathbf{r}(\mathbf{a},t)}{J(\mathbf{a},t)},$$

Role of viscosity

and viscous dynamics:

$$\frac{\partial \boldsymbol{\Omega}_0}{\partial t} = -\nu \, \mathrm{curl}_a \left(\frac{\hat{g}}{J} \, \mathrm{curl}_a \left(\frac{\hat{g}}{J} \, \boldsymbol{\Omega}_\mathbf{0} \right) \right)$$

where ν is the viscosity coefficient and the metric tensor $g_{\alpha\beta} = \frac{\partial x_i}{\partial a_\alpha} \cdot \frac{\partial x_i}{\partial a_\beta}$ (Kuznetsov, 2002). These equations can be used for construction of caustic ($\nu \to 0$).

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THANKS