Formation of Singularities on the Interface of Dielectric Liquids in a Strong Vertical Electric Field

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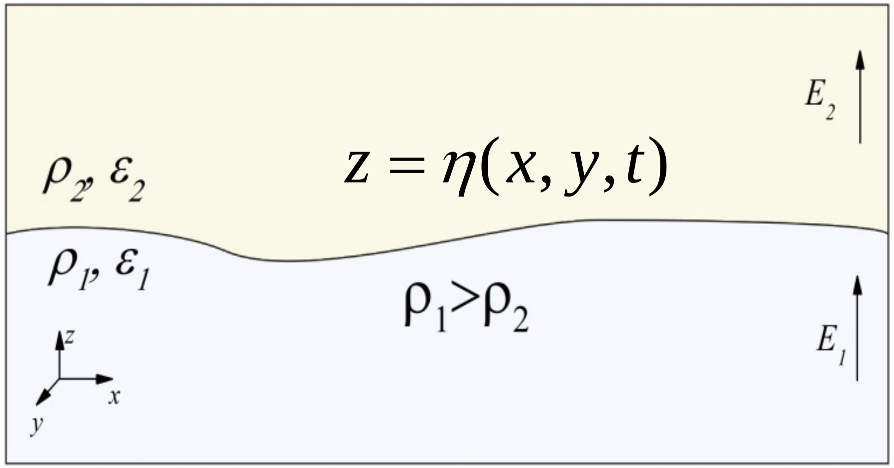
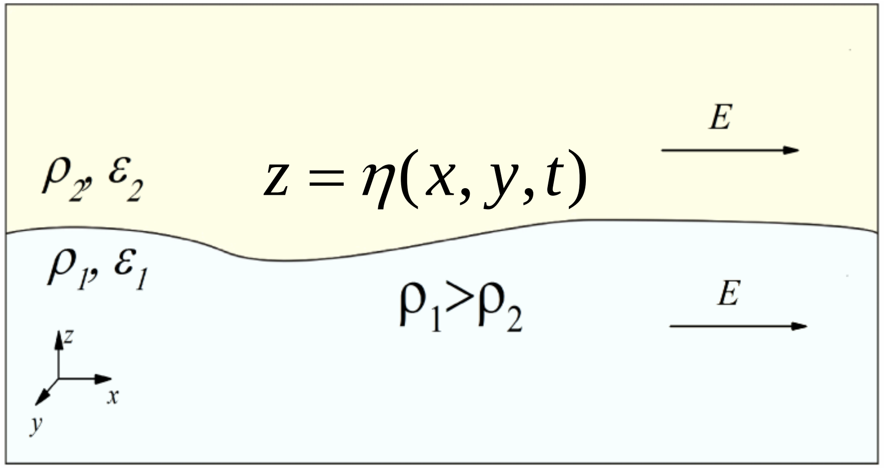
Vertical electric field.



The vertical electric field has a destabilizing effect on the interface of dielectric liquids.

Horizontal electric field.



As opposed to the vertical field, the horizontal electric field has a stabilizing effect on the interface.

We assume that both liquids are inviscid and incompressible, and the flow is irrotational (potential).

The functions $\Phi_{1,2}$, $\varphi_{1,2}$ are the velocity and electric field potentials.

Initial equations

$$\Delta\Phi_1 = 0, \qquad \Delta\varphi_1 = 0, \qquad z < \eta(x, y, t),$$

$$\Delta\Phi_2 = 0, \qquad \Delta\varphi_2 = 0, \qquad z > \eta(x, y, t),$$

$$\rho_1\left(\frac{\partial\Phi_1}{\partial t} + \frac{(\nabla\Phi_1)^2}{2}\right) - \rho_2\left(\frac{\partial\Phi_2}{\partial t} + \frac{(\nabla\Phi_2)^2}{2}\right) = \frac{\varepsilon_0(\varepsilon_2 - \varepsilon_1)}{2}\left(E_1E_2 - (\nabla\varphi_1\cdot\nabla\varphi_2)\right), \quad z = \eta(x, y, t),$$

$$\frac{\partial\eta}{\partial t} = \frac{\partial\Phi_1}{\partial z} - (\nabla_\perp\eta\cdot\nabla_\perp\Phi_1) = \frac{\partial\Phi_2}{\partial z} - (\nabla_\perp\eta\cdot\nabla_\perp\Phi_2), \qquad z = \eta(x, y, t),$$

$$\varphi_1 = \varphi_2, \qquad \varepsilon_1\left(\frac{\partial\varphi_1}{\partial z} - (\nabla_\perp\eta\cdot\nabla_\perp\varphi_1)\right) = \varepsilon_2\left(\frac{\partial\varphi_2}{\partial z} - (\nabla_\perp\eta\cdot\nabla_\perp\varphi_2)\right), \qquad z = \eta(x, y, t).$$

Conditions at infinity

1. Vertical electric field:

$$\Phi_{1,2} \to 0, \qquad z \to \mp\infty,$$

$$\varphi_{1,2} \to -E_{1,2}z, \qquad z \to \mp\infty,$$

$$\varepsilon_1E_1 = \varepsilon_2E_2.$$

2. Horizontal electric field:

$$\Phi_{1,2} \to 0, \qquad z \to \mp\infty,$$

$$\varphi_{1,2} \to -Ex, \qquad z \to \mp\infty,$$

$$E_1 = E_2 \equiv E.$$

Hamiltonian formalism

The equations of motion can be written in the Hamiltonian form [1,2]:

$$\psi_t = -\frac{\delta H}{\delta \eta}, \quad \eta_t = \frac{\delta H}{\delta \psi}.$$

where $\psi(x,y,t) = \rho_1 \phi_1 |_{z=\eta} - \rho_2 \phi_2 |_{z=\eta}$ and

$$H = \rho_1 \int\limits_{z\le\eta} \frac{(\nabla\Phi_1)^2}{2} d^3r + \rho_2 \int\limits_{z\ge\eta} \frac{(\nabla\Phi_2)^2}{2} d^3r$$
$$- \varepsilon_0\varepsilon_1 \int\limits_{z\le\eta} \frac{(\nabla\varphi_1)^2 - E_1^2}{2} d^3r - \varepsilon_0\varepsilon_2 \int\limits_{z\ge\eta} \frac{(\nabla\varphi_2)^2 - E_2^2}{2} d^3r.$$

[1]. V.E. Zakharov, Prikl. Mekh. Tekh. Fiz. 2, 86 (1968).
[2]. E.A. Kuznetsov, M.D. Spector, JETP 71, 22 (1976).

Vertical electric field; the small-angle approximation

Let us pass to dimensionless variables:

$$\psi \to \psi \frac{E_1}{k}\sqrt{\varepsilon_0 \varepsilon_1 \rho_1}, \quad \eta \to \frac{\eta}{k}, \quad t \to \frac{t}{E_1 k}\sqrt{\frac{\rho_1}{\varepsilon_0 \varepsilon_1}}, \quad r \to \frac{r}{k}.$$

We consider that $|\nabla_\perp \eta| \sim \alpha << 1.$

Expanding the integrand in the Hamiltonian in powers of the canonical variables up to the second- and third-order terms, we get:

$$H = \left(\frac{1+A}{4}\right)\int\left(\psi\hat{k}\psi - A\eta\left((\hat{k}\psi)^2 - (\nabla_\perp\psi)^2\right)\right)dxdy - \left(\frac{A_E^2}{1-A_E}\right)\int\left(\eta\hat{k}\eta + A_E\eta\left((\hat{k}\eta)^2 - (\nabla_\perp\eta)^2\right)\right)dxdy,$$

where $A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ is the Atwood number, and $A_E = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$ is its analog for the dielectric constants.

Here $\hat{k}f = -\frac{1}{2\pi}\iint \frac{f(x', y')}{\left[(x'-x)^2 - (y'-y)\right]^{3/2}} dx'dy', \qquad \hat{k}e^{ikr} = |k|e^{ikr}.$

Vertical electric field; the small-angle approximation

The equations of motion:

$$\psi_t - \left(\frac{2A_E^2}{1-A_E}\right)\hat{k}\eta = \frac{A(1+A)}{4}\Big((\hat{k}\psi)^2 - (\nabla_\perp\psi)^2\Big) + \frac{A_E^3}{1-A_E}\Big((\hat{k}\eta)^2 - (\nabla_\perp\eta)^2\Big) + \left(\frac{2A_E^3}{1-A_E}\right)\Big(\hat{k}(\eta\hat{k}\eta) + \nabla_\perp(\eta\nabla_\perp\eta)\Big),$$

$$\eta_t - \left(\frac{1+A}{2}\right)\hat{k}\psi = -\frac{A(1+A)}{2}\Big(\hat{k}(\eta\hat{k}\psi) + \nabla_\perp(\eta\nabla_\perp\psi)\Big).$$

Let us introduce the new functions $f = (c\psi + \eta)/2, \quad g = (c\psi - \eta)/2.$

The equations take the form

$$\tau f_t - \hat{k}f = \frac{(A+A_E)}{4}\Big[(\hat{k}f)^2 - (\nabla_\perp f)^2\Big] + \frac{(A-A_E)}{2}\Big[\hat{k}(f\hat{k}f) + \nabla_\perp(f\nabla_\perp f)\Big] + O(\alpha^3),$$

$$\tau g_t + \hat{k}g = \frac{(A+A_E)}{4}\Big[(\hat{k}f)^2 - (\nabla_\perp f)^2\Big] + \frac{(A+A_E)}{2}\Big[\hat{k}(f\,\hat{k}f) + \nabla_\perp(f\nabla_\perp f)\Big] + O(\alpha^3).$$

Here $c = \dfrac{\sqrt{(1-A_E)(1+A)}}{2\,|A_E|}, \quad \tau = \dfrac{1}{|A_E|}\sqrt{\dfrac{1-A_E}{1+A}}.$

Two special cases:

1. $A_E = +A \quad \Leftrightarrow \quad \rho_1/\rho_2 = \varepsilon_1/\varepsilon_2,$

2. $A_E = -A \quad \Leftrightarrow \quad \rho_1/\rho_2 = \varepsilon_2/\varepsilon_1.$

Some pairs of immiscible dielectric liquids

Lower fluid	ε_1	$\rho_1,\ kg/m^3$	Upper fluid	ε_2	$\rho_2,\ kg/m^3$	A_E	A
PMPS	2.7	1100	spindle oil	1.9	870	0.17	0.12
PMPS	2.7	1100	linseed oil	3.2	930	-0.084	0.084
LH	1.05	125	vacuum	1	0	-1 (formally)	1
water	81	1000	air	1	1	0.98	1

Here PMPS is liquid organosilicon polymer, the polymethylphenylsiloxane; LH is liquid helium with the free surface charged by the electrons [3,4].

The conditions $A_E = A$, or $A_E = -A$ are satisfied with acceptable accuracy for these pairs.

[3]. N.M. Zubarev, JETP Lett. **71**, 367 (2000).
[4]. N.M. Zubarev, JETP **94**, 534 (2002).

Dynamics of the interface for the case $A_E = A$

The equations of motion (compare with Refs. [5,6]):

$$\tau f_t - \hat{k} f = \frac{A}{2}\left[(\hat{k} f)^2 - (\nabla_\perp f)^2\right],$$

$$\tau g_t + \hat{k} g = \frac{A}{2}\left[(\hat{k} f)^2 - (\nabla_\perp f)^2\right] + A\left[\hat{k}(f\hat{k} f) + \nabla_\perp(f\nabla_\perp f)\right].$$

<u>2D geometry:</u> $\hat{k} = -\hat{H}\frac{\partial}{\partial x}, \quad \hat{H}\phi(x) = \frac{1}{\pi} p.v.\int_{-\infty}^{+\infty} \frac{\phi(x')}{x - x'} dx',$ where $\hat{H}$ is Hilbert transform.

The equations take the following form:

$$\tau F_t + iF_x = -AF_x^2,$$

$$\tau G_t - iG_x = -AF_x^2 + 2A\hat{P}\left(F\bar{F}_x\right)_x.$$

Here $F = \hat{P} f, \quad G = \hat{P} g,$ where $\hat{P} = (1 - i\hat{H})/2$ is the projection operator. These functions are analytical in the upper half-plane of the complex variable x.

[5]. E.A. Kuznetsov, M.D. Spector, and V.E. Zakharov, Phys. Rev. E **49**, 1283 (1994).
[6]. N.M. Zubarev, JETP **114,** 2043 (1998).

Dynamics of the interface for the case $A_E = A$

The equation on F transforms to the complex Hopf equation:

$$\tau V_t + iV_x = -2AVV_x, \qquad V = F_x.$$

Its solution has the form:

$$V = V_0(x'), \qquad x\tau = x'\tau + it + 2AV_0(x')t, \qquad V_0(x) = V|_{t=0},$$

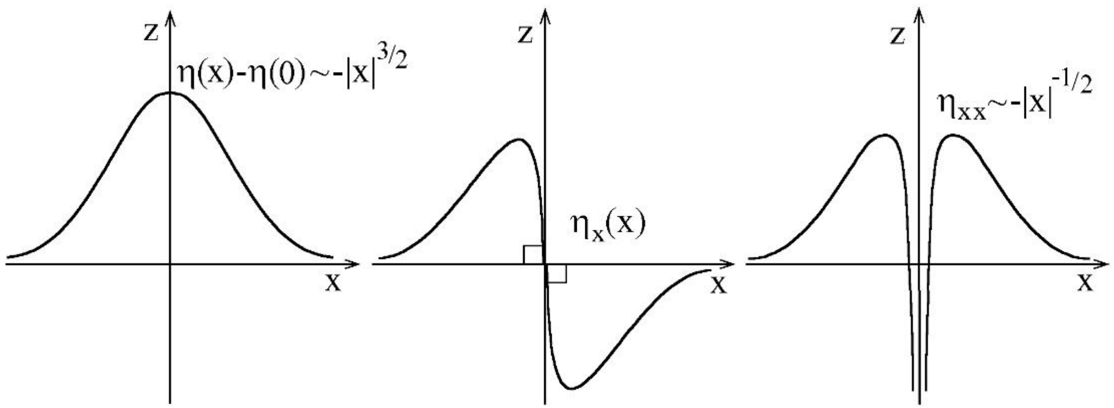
The equation on G can be also solved:

$$G = \frac{1}{\tau}\int_0^t Q(x + it/\tau - it'/\tau, t')dt', \qquad Q(x,t) = -AF_x^2 + 2A\hat{P}\left(F\overline{F}_x\right)_x.$$

Weak root singularities are formed at the interface: $z - z_c \sim -|x - x_c|^{3/2}$.

For these singularities the curvature becomes infinite in a finite time, and the boundary remains smooth:

$$\eta_x \sim -(x - x_c)\cdot|x - x_c|^{-1/2},$$
$$\eta_{xx}(x,t_c) \sim -|x - x_c|^{-1/2},$$
$$\eta_{xx}(x_c,t) \sim -(t_c - t)^{-1/2}.$$



Dynamics of the interface for the case $A_E = -A$

The equations of motion:

$$\tau f_t - \hat{k} f = A\left[\hat{k}(f\hat{k}f) + \nabla_\perp (f \nabla_\perp f)\right],$$
$$\tau g_t + \hat{k} g = 0.$$

According to the second equation, $g \to 0.$

As a consequence, we can put $\eta = \dfrac{(1+A)}{2A}\psi.$

The equation of interface motion in 2D geometry:

$$\eta_t + A\hat{H}\eta_x = A^2\left[\hat{H}\left(\eta \hat{H}\eta_x\right)_x + \left(\eta\eta_x\right)_x\right].$$

It can be rewritten as $F_t + iAF_x = 2A^2\hat{P}\left(F\bar{F}_x\right)_x,$ where $F = \hat{P}\eta.$

This integro-differential equation can be reduced to the set of ordinary differential equations by the substitution:

$$F(x,t) = \sum_{n=1}^{N} \frac{iS_n/2}{x + p_n(t)}, \qquad \frac{dp_n}{dt} = -iA + iA^2 \sum_{j=1}^{N} \frac{S_j}{(p_n - \bar{p}_j)^2}, \qquad n = 1, 2, \ldots, N.$$

Dynamics of the interface for the case $A_E = -A$

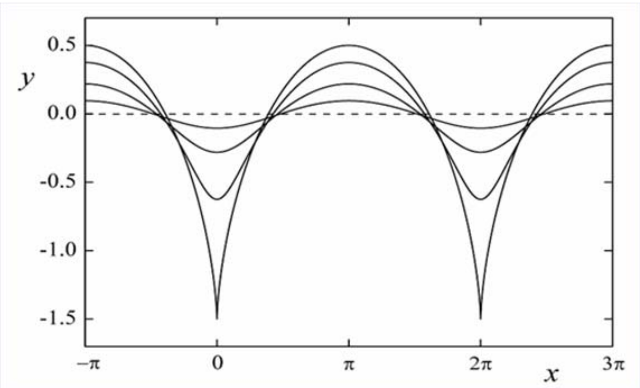
Exact particular solution for $N = 1$:

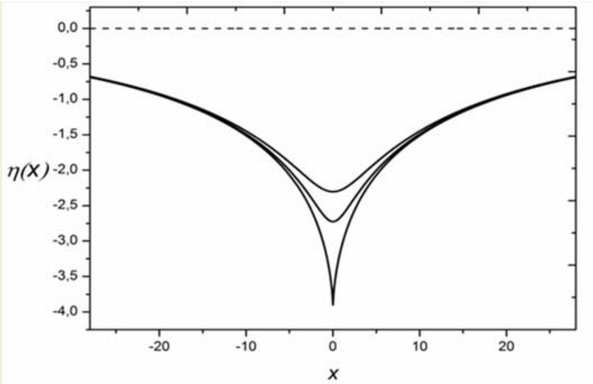
$$\eta(x,t) = \frac{Sa(t)}{x^2 + a^2(t)}, \qquad \text{where} \qquad \begin{aligned} & a(t) + \frac{\sqrt{AS}}{4}\ln\left(\frac{2a(t)-\sqrt{AS}}{2a(t)+\sqrt{AS}}\right) = A(t_0 - t), \qquad S > 0, \\ & a(t) + \frac{\sqrt{A|S|}}{2}\arctan\left(\frac{2a(t)}{\sqrt{A|S|}}\right) = A(t_c - t), \qquad S < 0. \end{aligned}$$

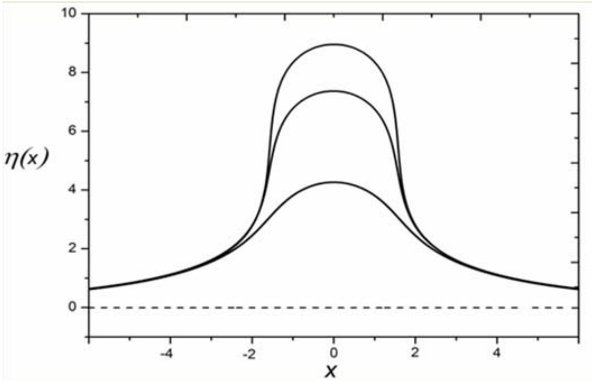
The boundary shape becomes singular at some moment $t = t_c$:

$$\eta(x,t_c) = \lim_{a\to 0}\left(\frac{Sa}{x^2 + a^2}\right) = \pi S\delta(x), \qquad S < 0.$$

In the formal limit $A \to 1$, $\rho_2/\rho_1 \to 0$ the equation of motion is reduced to the Laplace Growth Equation [7]. It describes the formation of cusps at the interface in a finite time, or the formation of so-called “fingers” (see figures).







[7]. N.M. Zubarev, Phys. Fluids **18**, art. no. 028103 (2006).

Horizontal electric field; the small-angle approximation

We consider that $|\nabla_\perp \eta| \sim \alpha << 1.$

The Hamiltonian takes the following form:

$$H = \left(\frac{1+A}{4}\right)\iint\left[\psi\hat{k}\psi - A\eta\left((\hat{k}\psi)^2 - (\nabla_\perp\psi)^2\right)\right]dxdy$$
$$+\left(\frac{A_E^2}{1+A_E}\right)\iint\left[\eta_x\hat{k}^{-1}\eta_x + A_E\left(\eta\eta_x^2 - \eta_x\hat{k}^{-1}\eta\hat{k}\eta_x + \eta_x(\nabla_\perp\eta\cdot\nabla_\perp\hat{k}^{-1}\eta_x)\right)\right]dxdy.$$

The equations of motion:

$$\psi_t - \left(\frac{2A_E^2}{1+A_E}\right)\hat{k}^{-1}\eta_{xx} = \frac{A(1+A)}{4}\left[(\hat{k}\psi)^2 - (\nabla_\perp\psi)^2\right]$$
$$+\left(\frac{A_E^2}{1+A_E}\right)\left[\eta_x^2 + 2\eta\eta_{xx} + (\nabla_\perp\hat{k}^{-1}\eta_x)^2 - \hat{k}^{-1}\partial_x\left(\eta\hat{\eta}_x - \nabla_\perp\eta\cdot\nabla_\perp\hat{k}^{-1}\eta_x\right)\right],$$

$$\eta_t - \left(\frac{1+A}{2}\right)\hat{k}\psi = -\frac{A(1+A)}{2}\left[\hat{k}(\eta\hat{k}\psi) + \nabla_\perp(\eta\nabla_\perp\eta)\right].$$

Dynamics of the interface

The equation for the interface evolution:

$$\eta_{tt} - v_0^2\eta_{xx} = \frac{\hat{k}}{2}\left(v_0^2 A_E \eta_x^2 - A\eta_t^2\right) + \hat{k}\left(v_0^2 A_E \eta\eta_{xx} - A\eta\eta_{tt}\right) + \frac{\hat{k}}{2}\left[v_0^2 A_E (\nabla_\perp \eta_x)^2 - A(\nabla_\perp \eta_t)^2\right]$$
$$+\nabla_\perp\left[v_0^2 A_E \partial_x(\eta\nabla_\perp \hat{k}^{-1}\eta_x) - A\partial_t(\eta\nabla_\perp \hat{k}^{-1}\eta_t)\right].$$

Here $v_0 = A_E\sqrt{(1+A)/(1+A_E)}$ is the velocity of linear waves.

Linear approximation: $\eta_{tt} = v_0^2\eta_{xx}.$

General solution of the linear wave equation:

$$\eta(x,y,t) = f(x - v_0 t, y) + g(x + v_0 t, y).$$

Lets us consider the special case $A = A_E.$

The equation of motion admits two exact particular solutions:

$$\eta(x,y,t) = f(x - At, y),$$
$$\eta(x,y,t) = g(x + At, y).$$

Interaction of counter-propagating waves

The approximate solution of the equations of motion has the form:

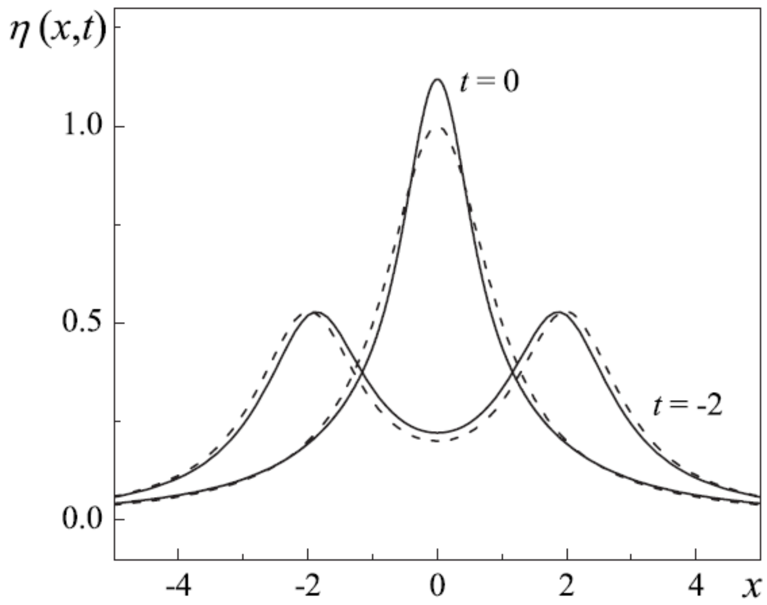
$$\eta(x,y,t) = f(x - At, y) + g(x + At, y) - \frac{A}{2}\hat{k}(fg + \nabla_\perp \hat{k}^{-1} f \cdot \nabla_\perp \hat{k}^{-1} g)$$
$$- \frac{A}{2}\nabla_\perp (f \nabla_\perp \hat{k}^{-1} g + g \nabla_\perp \hat{k}^{-1} f) + O(\alpha^3).$$

This formula describes the nonlinear superposition of the oppositely directed waves.

Interaction of the plane solitary waves:

$$f(x) = g(x) = \frac{0.5}{(1 + x^2)}$$

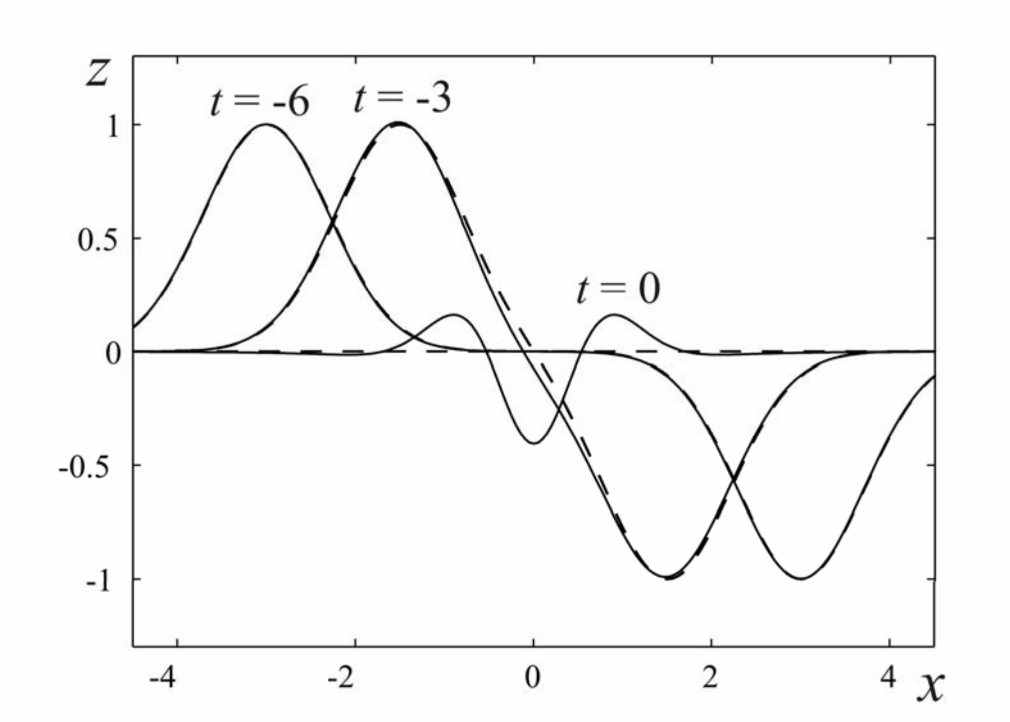
$$A = A_E = 1$$

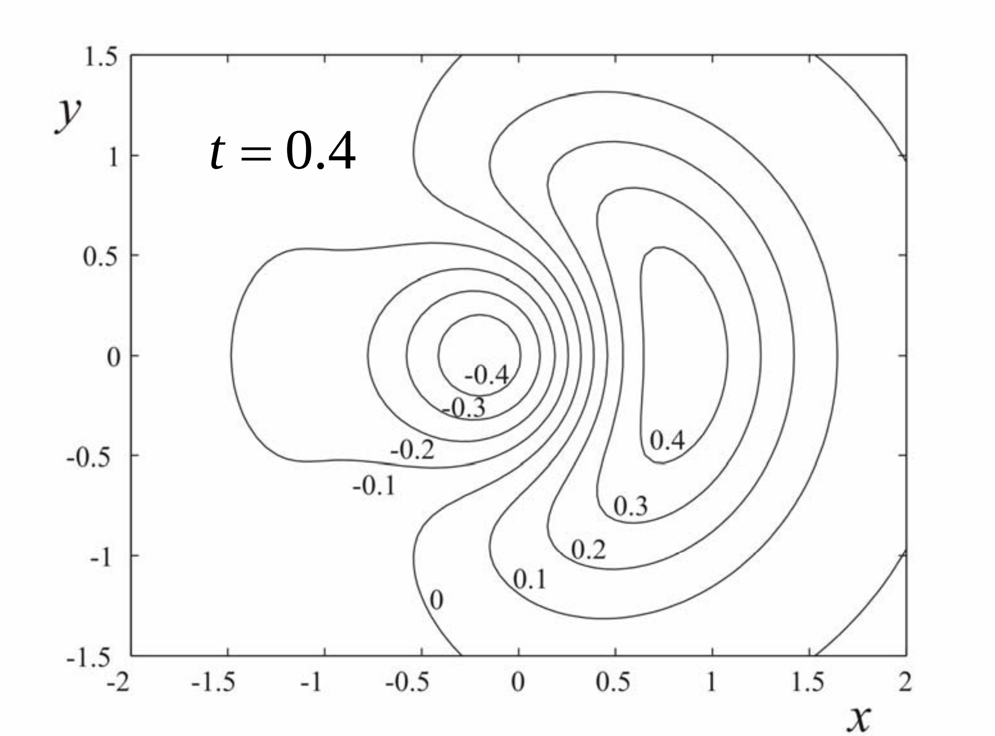


Interaction of two 3D solitary waves

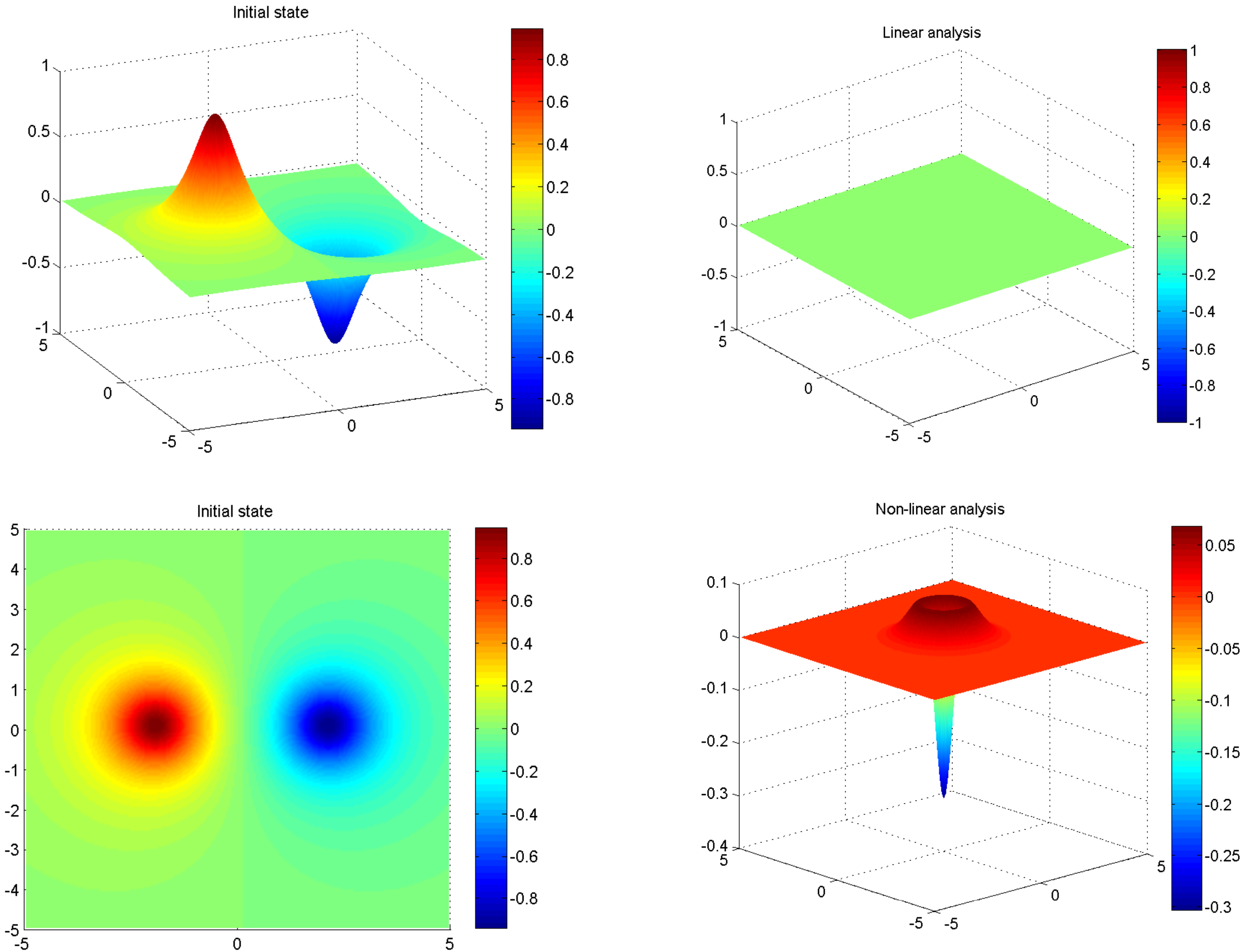
$$A = A_E = 0.5$$

$$f(x, y) = \exp(-x^2 - y^2), \qquad g(x, y) = -\exp(-x^2 - y^2).$$

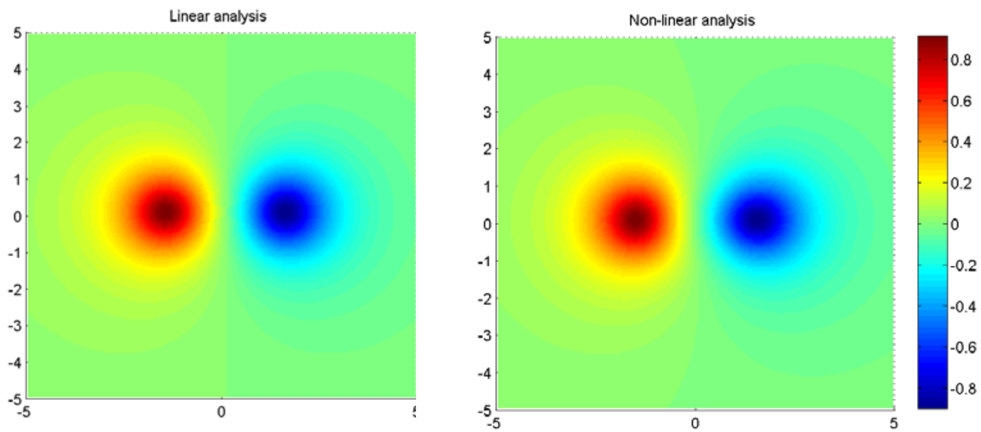




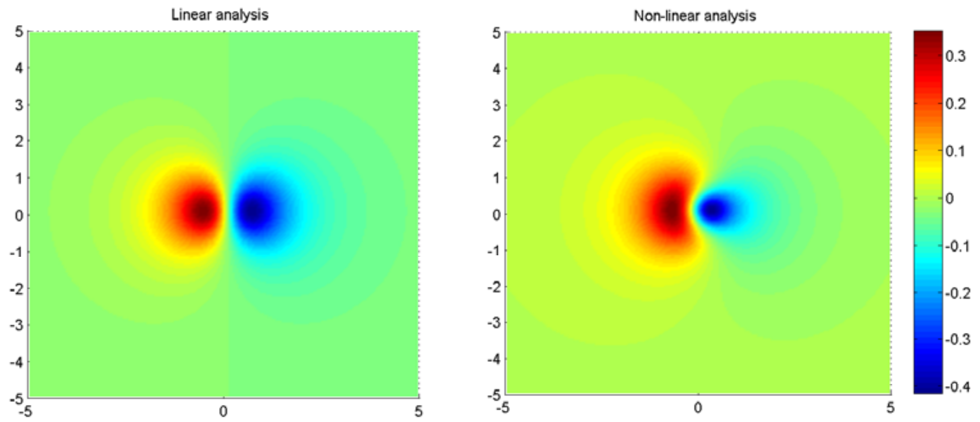
Interaction of two 3D solitary waves



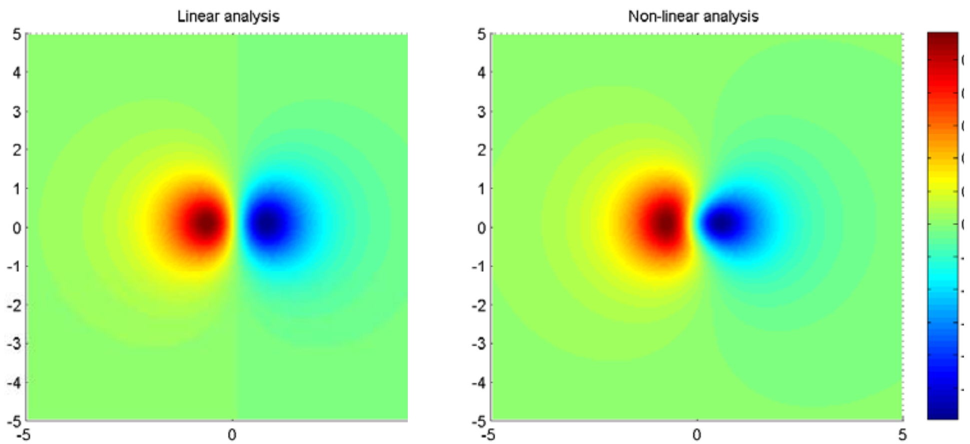
Interaction of two 3D solitary waves



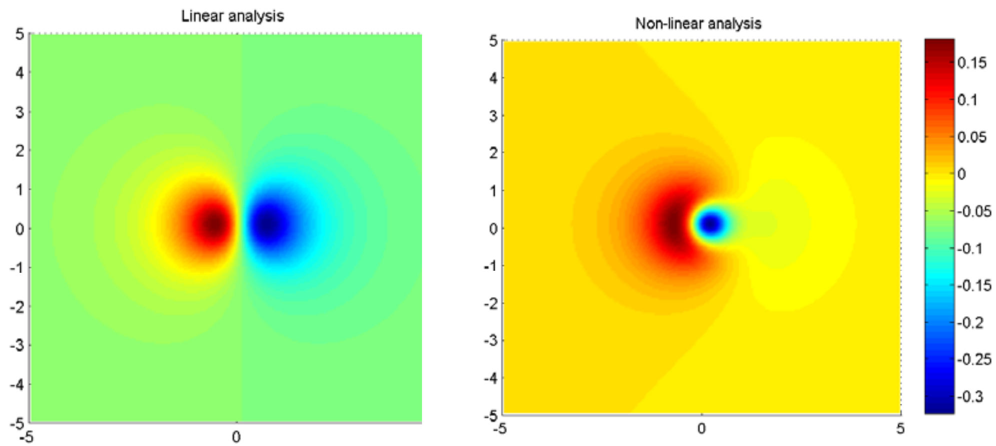
t=-1.5



t=-0.25



t=-0.5



t=-0.1

Conclusion

The nonlinear dynamics of the interface between two ideal dielectric liquids in an external electric field was considered. A number of particular cases, where the evolution of the interface can be effectively studied analytically, were revealed.

Vertical electric field (the interface is unstable)		Horizontal electric field (the interface is linearly stable)	
$A_E = A$	$A_E = -A$	$A_E = A$	$A_E = -A$
Formation of weak (root) singularities at the interface	Formation of strong singularities (cusps) at the interface	Nondispersive propagation of weakly nonlinear waves	The problem becomes intergrable if upper fluid moves relative to the lower one
The small angle approximation is valid	The small angle approximation is violated	The small angle approximation is valid	The equations admit nonsingular exact solutions

Thank you
for attention!