Recursion operator for the Narita-Itoh-Bogoyavlensky lattice

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Evolutionary differential-difference equations

$$u_t = K(u_q, u_{q+1}, \cdots u_p), \quad q, p \in \mathbb{Z}, \quad q \leq j \leq p$$
$$u_t = \partial_t u, \quad u_j = \mathcal{S}^j u(n,t) = u(n+j,t)$$

The order of K is (q,p) is $\partial_{u_q} K \partial_{u_p} K \neq 0$ and its total order $p-q$.

The Volterra Chain

$$u_t = u(u_1 - u_{-1})$$

is of order $(-1,1)$ with total order 2.

Motivations

- Integrable discretisation of integrable systems

Example. The equation

$$u_t = u^2(u_2 u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1})$$

is of oder (-2,2) and it can be interpreted as the Sawada-Kotera equation

$$U_\tau = U_{xxxxx} + 5UU_{xxx} + 5U_x U_{xx} + 5U^2 U_x$$

under the following continuous limit at $\epsilon \to 0$:

$$u(n,t) = \frac{1}{3} + \frac{\epsilon^2}{9} U(x - \frac{4}{9}\epsilon t, \tau + \frac{2\epsilon^5}{135} t), \quad x = \epsilon n.$$

(Alder: arXiv:11035139)

• Generalised symmetry of discrete equations

Example. The discrete Korteweg-de Vries equation

$$(u_{1,1} - u_{0,0})(u_{1,0} - u_{0,1}) = \alpha - \beta$$

possesses a generalised symmetry of order $(-1, 1)$:

$$u_\tau = \frac{1}{u_{1,0} - u_{-1,0}}.$$

This can be transformed into the modified Volterra chain

$$v_\tau = v^2(v_1 - v_{-1}),$$

where $v = \frac{1}{u_{1,0} - u_{-1,0}}$.

- Classification problems are still open

The following types have been classified:

1. Volterra type: $u_t = f(u_{-1}, u, u_1)$;

2. Toda type: $u_{tt} = f(u_t, u_{-1}, u, u_1)$;

3. Relativistic Toda-Type:

$$u_t = f(u_1, u, v), v_t = g(v_{-1}, v, u)$$

and

$$u_{tt} = f(u_1, u, u_{1,t}, u_t) - g(u, u_{-1}, u_t, u_{-1,t})$$

Complex of variational calculus

$U_{\mathrm{s}} = \{u_n \,|\, n \in \mathbb{Z}\}$
$\mathcal{F}_{\mathrm{s}} = \{$smooth functions of variables $U_{\mathrm{s}}\}$
$[g]$ an equivalent class: $g \equiv h \Leftrightarrow g - h \in \mathrm{Im}\,\Delta$, $\Delta = \mathcal{S} - 1$;
$\mathcal{F}'_{\mathrm{s}}$: the space of equivalent classes
Lie algebra $\mathfrak{h}$: the space of evolutionary vector fields.

$$\partial = \sum_{k\in\mathbb{Z}} h_k \cdot \frac{\partial}{\partial u_k} \stackrel{[\partial,\mathcal{S}]=0}{\longrightarrow} \partial_P = \sum_{k\in\mathbb{Z}} \mathcal{S}^k P \cdot \frac{\partial}{\partial u_k} \Longrightarrow \mathfrak{h}$$

$\mathcal{F}'_{\mathrm{s}}$ is a $\mathfrak{h}$-module with a representation as follows:

$$P \circ g = [\partial_P(g)] = [\sum_{k\in\mathbb{Z}} (\mathcal{S}^k P) \frac{\partial g}{\partial u_k}], \;\; P \in \mathfrak{h}, \;\; g \in \mathcal{F}'_{\mathrm{s}}$$

What is the space Ω^n?

$\Omega^0 = \mathcal{F}'_{\mathrm{s}}$

A natural non-degenerate pairing between ∂_P and a vertical 1-form $\omega = \sum_k h^k \cdot \mathrm{d}u_k$:

$$< \omega,\ P >= [\sum_{n\in\mathbb{Z}} h^{(n)}\mathcal{S}^n P] =< \sum_{n\in\mathbb{Z}} \mathcal{S}^{-n}h^{(n)},\ P > .$$

$$\omega \to \xi \cdot \mathrm{d}u, \quad \xi = \textstyle\sum_n \mathcal{S}^{-n}h^{(n)}\mathrm{d}u_0 \Longrightarrow \Omega^1$$

$$\mathrm{d} : \Omega^0 \to \Omega^1 \Longrightarrow \delta(g) = \sum_k \mathcal{S}^{-k}\frac{\partial g}{\partial u_k}$$

Fréchet derivatives and Lie derivatives

Def. For any objects in the complex $\mathcal{O}$, its Fréchet derivative along a vector field $P \in \mathfrak{h}$ is defined as

$$D_{\mathcal{O}}[P] = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \mathcal{O}[u + \epsilon P].$$

Eg. For $\mathcal{H} = u(\mathcal{S} - \mathcal{S}^{-1})u$,
$D_{\mathcal{H}}[P] = P(\mathcal{S} - \mathcal{S}^{-1})u + u(\mathcal{S} - \mathcal{S}^{-1})P$.

Thm. Let L_K denote Lie derivative along $K \in \mathfrak{h}$. Then

$L_K g = [D_g[K]] \in \mathcal{F}'_{\mathbf{s}}$ for $g \in \mathcal{F}'_{\mathbf{s}}$; → conserved density
$L_K h = [K, h]$ for $h \in \mathfrak{h}$; → symmetry
$L_K \xi = D_\xi[K] + D_K^\star(\xi)$ for $\xi \in \Omega^1$; → cosymmetry
$L_K \mathcal{R} = D_{\mathcal{R}}[K] - D_K \mathcal{R} + \mathcal{R} D_K$ for $\mathcal{R} : \mathfrak{h} \to \mathfrak{h}$; → recursion Op.
$L_K \mathcal{H} = D_{\mathcal{H}}[K] - D_K \mathcal{H} - \mathcal{H} D_K^\star$ for $\mathcal{H} : \Omega^1 \to \mathfrak{h}$; → Hamiltonian
$L_K \mathcal{I} = D_{\mathcal{I}}[K] + D_K^\star \mathcal{I} + \mathcal{I} D_K$ for $\mathcal{I} : \mathfrak{h} \to \Omega^1$. → symplectic

All results related about concepts for evolutionary partial differential equations are **valid** for evolutionary differential-difference equations.

A recursion operator of Volterra chain

$$\Re = u\mathcal{S} + u + u_1 + u\mathcal{S}^{-1} + u_t(\mathcal{S} - 1)^{-1}\frac{1}{u}$$

generating local symmetries of order $(-n, n)$, e.g.

$$u_{t_1} = u(u_1 - u_{-1})$$
$$u_{t_2} = uu_1(u + u_1 + u_2) - u_{-1}u(u_{-2} + u_{-1} + u)$$
$$\cdots\cdots$$

Conservation laws

A pair of functions $(\rho,\ \sigma)$ is called a conservation law of an equation $u_t = K$ if

$$D_t\rho = (\mathcal{S} - 1)\sigma\Big|_{u_t=K}.$$

The functions ρ and σ are called the density and flux of the conservation law respectively.

The Volterra chain

$$\begin{aligned}
&u_t = (\mathcal{S} - 1)\left(uu_{-1}\right)\\
&\partial_t \ln u = \frac{u_t}{u} = u_1 - u_{-1} = (\mathcal{S} - 1)\left(u + u_{-1}\right)\\
&\cdots\cdots
\end{aligned}$$

Residues and Adler's Theorem

Consider Laurent formal difference series of order N

$$A = a^N \mathcal{S}^N + a^{N-1} \mathcal{S}^{N-1} \dots$$

The residue res(A) and the logarithmic residue res ln(A) are defined as

$$\text{res}(A) = a^0, \qquad \text{res}\ln(A) = \ln(a^N)\,.$$

Adler's Theorem Let A and B be two Laurent formal difference series of order N and M respectively. Then

$$\text{res}[A, B] = (\mathcal{S} - \mathbf{1})(\sigma(A, B)),$$

where

$$\sigma(A, B) = \sum_{i=1}^{N} \sum_{k=1}^{i} \mathcal{S}^{-k}(a^{-i}) \mathcal{S}^{i-k}(b^i) - \sum_{i=1}^{M} \sum_{k=1}^{i} \mathcal{S}^{-k}(b^{-i}) \mathcal{S}^{i-k}(a^i).$$

Infinitely many conserved densities

Thm. Consider an equation $u_t = K$. If there exists a series $\Re_L$ such that

$$D_{\Re_L}[K] = [D_K,\ \Re_L],$$

$\mathrm{res}(\Re_L^i)$ and $\mathrm{res}\ln(\Re_L)$ are its conserved densities.

The Volterra chain

$$\Re_L = u\mathcal{S} + u + u_1 + u\mathcal{S}^{-1} + \sum_{i=1} \frac{u_t}{u_{-i}}\mathcal{S}^{-i}$$

$$\begin{aligned}
&\rho_0 = \mathrm{res}\ln(\Re_L) = \ln u \\
&\rho_1 \mathrm{res}(\Re_L) = u + u_1 \equiv 2u \\
&\rho_2 = \mathrm{res}(\Re_L^2) = 3uu_1 + u_1u_2 + u^2 + u_1^2 \equiv 4uu_1 + 2u^2 \\
&\Big(D_t\rho_2 = 4(\mathcal{S}-1)(u^2u_{-1} + u_{-1}uu_1)\Big) \\
&\cdots\cdots
\end{aligned}$$

Bi-Hamiltonian structures

$$u_t = \mathcal{H}_1 \delta_u f = \mathcal{H}_2 \delta_u g,$$

where $\mathcal{H}_1$, $\mathcal{H}_2$ are Hamiltonian operators and δ_u is the variational derivative.

The Volterra chain

$$u_t = \mathcal{H}_1 \delta_u u = \mathcal{H}_2 \delta_u \frac{\ln u}{2},$$

$$\mathcal{H}_1 = u(\mathcal{S} - \mathcal{S}^{-1})u,$$
$$\mathcal{H}_2 = \Re \mathcal{H}_1 = u(1 + \mathcal{S}^{-1})(\mathcal{S}u - u\mathcal{S}^{-1})(1 + \mathcal{S})u \ .$$

Narita-Itoh-Bogoyavlensky lattices (1980's): $p \in \mathbb{N}$

$$u_t = u\left(\sum_{k=1}^{p} u_k - \sum_{k=1}^{p} u_{-k}\right);$$

$$v_t = v\left(\prod_{k=1}^{p} v_k - \prod_{k=1}^{p} v_{-k}\right);$$

$$w_t = w^2\left(\prod_{k=1}^{p} w_k - \prod_{k=1}^{p} w_{-k}\right).$$

$$u = \prod_{k=0}^{p-1} v_k \quad \text{and} \quad u = \prod_{k=0}^{p} w_k.$$

For finite lattices, work has been done on Hamiltonian structures, associations with classical Lie algebras and the r-matrix structure etc (Suris, Nijhoff, Papageorgiou...).

Discrete Sawada-Kotera equation (dSK) (Alder: arXiv:11035139):

$$u_t = u^2(u_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1})$$

- Tsujimoto and Hirota (1996): continuous limit of the reduced discrete BKP hierarchy.
- Both $u_{t'} = u(u_1 - u_{-1})$ and $u_{t''} = u^2(u_2u_1 - u_{-1}u_{-2})$ are integrable, but do not commute.
- Lax representation: $L = (\mathcal{S} + u)^{-1}(u\mathcal{S} + 1)\mathcal{S}^2$ $A = (u_{-1}\mathcal{S} + 1 - u_{-1}u_{-2} + u_{-2}\mathcal{S}^{-1})(\mathcal{S} - \mathcal{S}^{-1})$.

Symmetries of dSK: $u_t := P^4 + P^2$

$$
\begin{aligned}
& u^2(u_1u_2^2u_3u_4 + u_1^2u_2^2u_3 + uu_1^2u_2^2 + u_{-1}uu_1^2u_2 \\
& -u_{-2}u_{-1}^2uu_1 - u_{-2}^2u_{-1}^2u - u_{-3}u_{-2}^2u_{-1}^2 - u_{-4}u_{-3}u_{-2}^2u_{-1}) \\
& + \cdots + u(u_1u_2 + u_1^2 + u_1u - uu_{-1} - u_{-1}^2 - u_{-1}u_{-2}) \\
=: {} & Q^7 + Q^5 + Q^3 \\
\Longrightarrow {} & [P^4,\ Q^7] = 0; \qquad [P^2,\ Q^3] = 0.
\end{aligned}
$$

Cosymmetries: $G_1 = \frac{1}{u}, G_2 = u_1u_2 + u_1u_{-1} + u_{-1}u_{-2} - 1$

Questions: Hamiltonian strictures? Recursion operators?

The hierarchy dSK (Alder & Postnikov: arXiv:1107.2305)

$$
u_t = u^2\left(\prod_{i=1}^{p} u_i - \prod_{i=1}^{p} u_{-i}\right) - u\left(\prod_{i=1}^{p-1} u_i - \prod_{i=1}^{p-1} u_{-i}\right)
$$

What was known?

- $p = 1$: The Volterra chain

- $p = 2$: Zhang, Tu, Oevel & Fuchssteiner (1991)

$$\begin{aligned} u_t &= u(u_2 + u_1 - u_{-1} - u_{-2}) \\ &= u(\mathcal{S}^2 + \mathcal{S} - \mathcal{S}^{-1} - \mathcal{S}^{-2})u\delta_u u \end{aligned}$$

  has a recursion operator

$$\Re = u(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})(\mathcal{S}^2 u - u\mathcal{S}^{-1})(u\mathcal{S}^{-1} - \mathcal{S}u)^{-1} \\ (u\mathcal{S}^{-2} - \mathcal{S}u)(1 - \mathcal{S}^{-2})^{-1}u^{-1}$$

- For arbitrary p, the equation is Hamiltonian:

$$u_t = u(\sum_{k=1}^{p} \mathcal{S}^k - \sum_{k=1}^{p} \mathcal{S}^{-k})u\delta_u u.$$

Main Results

Thm. For any $p \in \mathbb{N}$, a recursion operator of the Narita-Itoh-Bogoyavlensky lattice is

$$\Re = u(\sum_{i=0}^{p} \mathcal{S}^{-i}) \prod_{i=1}^{\rightarrow p} (\mathcal{S}^{p+1-i}u - u\mathcal{S}^{-i})(\mathcal{S}^{p-i}u - u\mathcal{S}^{-i})^{-1} \ .$$

It is a Hamiltonian equation with respect to

$$\Re\mathcal{H} = u(\sum_{i=0}^{p} \mathcal{S}^{-i}) \left(\prod_{i=1}^{\rightarrow (p-1)} (\mathcal{S}^{p+1-i}u - u\mathcal{S}^{-i})(\mathcal{S}^{p-i}u - u\mathcal{S}^{-i})^{-1} \right)$$

$$(\mathcal{S}u - u\mathcal{S}^{-p})(\sum_{i=0}^{p} \mathcal{S}^{i})u \ ,$$

where $\mathcal{H} = u(\sum_{k=1}^{p} \mathcal{S}^{k} - \sum_{k=1}^{p} \mathcal{S}^{-k})u$. Indeed,

$$u_t = \frac{1}{p+1} \ \mathcal{H}\delta_u \ln u \ .$$

Example. When $p = 2$, the equation is bi-Hamiltonian.

$$
\begin{aligned}
u_t &= u(u_2 + u_1 - u_{-1} - u_{-2}) = u(\mathcal{S}^2 + \mathcal{S} - \mathcal{S}^{-1} - \mathcal{S}^{-2})u\delta_u u \\
&= \frac{u}{3}(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})(\mathcal{S}^2 u - u\mathcal{S}^{-1})(\mathcal{S}u - u\mathcal{S}^{-1})^{-1} \\
&\quad (\mathcal{S}u - u\mathcal{S}^{-2})(1 + \mathcal{S} + \mathcal{S}^2)u\delta_u \ln u
\end{aligned}
$$

$$
\begin{aligned}
&= u(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})(\mathcal{S}^2 u - u\mathcal{S}^{-1})(\mathcal{S}u - u\mathcal{S}^{-1})^{-1}(u_1 - u) \\
&= u(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})(u_2 - u) \\
&= u(u_2 - u + u_1 - u_{-1} + u - u_{-2})
\end{aligned}
$$

Lax representation for Bogoyavlensky hierarchy

$$L = \mathcal{S} + u\mathcal{S}^{-p}, \qquad B^{(n)} = (L^{(p+1)n})_{\geq 0}$$
$$L_{t_n} = [B^{(n)},\ L].$$

Idea to construct a recursion operator: (Tu ('89); Gürses, Karasu & Sokolov ('99))

1. Relate the difference operators $B^{(n)}$:

$$B^{(n+1)} = LB^{(n)} + R$$

with R is the reminder.
2. Find the relation between two flows corresponding to these two difference operators.

Construction of recursion operators I

$$L = \lambda U^{(0)} + U^{(1)}$$

The only non-zero entry is $(U^{(0)})_{11} = 1$.

$$U^{(1)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -u \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}_{(p+1)\times(p+1)}$$

Take ansatz

$$B^{(n+1)} = \lambda^{p+1} B^{(n)} + W,$$

$$W = \sum_{i=0}^{p+1} \lambda^{p+1-i} A^{(i)}, \quad A^{(i)} = (a_{kl}^{(i)})_{(p+1)\times(p+1)}$$

$$a_{j+i,j}^{(i)} \neq 0, \quad 1 \leq j \leq p+1, \quad i+j \equiv (i+j) \mod (p+1).$$

Reduction group $\mathbb{Z}_{p+1}$ of L

$$s : W(\lambda) \mapsto SW(\sigma\lambda)S^{-1}, \quad \omega = e^{2\pi i/(p+1)},$$

where S is a diagonal matrix with entries $S_{ii} = \sigma^i$.

The ansatz W is invariant under s. Clearly $s^{p+1} = id$.

The formula for computing the recursion operator:

$$L_{t_{n+1}} = \lambda^{p+1} L_{t_n} + \mathcal{S}(W)L - LW.$$

Construction of recursion operators II

$$\lambda^{p+2}: \quad \mathcal{S}(A^{(0)})U^{(0)} - U^{(0)}A^{(0)} = 0;$$

$$\lambda^{p+1}: \quad U^{(1)}_{t_n} + \mathcal{S}(A^{(1)})U^{(0)} - U^{(0)}A^{(1)} + \mathcal{S}(A^{(0)})U^{(1)} - U^{(1)}A^{(0)} = 0;$$

$$\lambda^{p+1-i}: \quad \mathcal{S}(A^{(i+1)})U^{(0)} - U^{(0)}A^{(i+1)} + \mathcal{S}(A^{(i)})U^{(1)} - U^{(1)}A^{(i)} = 0, \quad 1 \le i \le p;$$

$$\lambda^{0}: \quad U^{(1)}_{t_{n+1}} = \mathcal{S}(A^{(p+1)})U^{(1)} - U^{(1)}A^{(p+1)}.$$

Construction of recursion operators III

Important step: To establish the relation between $a^{(i+1)}_{i+2,1}$ and $a^{(i)}_{i+1,1}$ for $1 \leq i \leq p-1$.

$$a^{(i+1)}_{i+2,1} = \mathcal{S}^{-1}(\mathcal{S}^i u - u\mathcal{S}^{i-p})^{-1}(\mathcal{S}^i u - u\mathcal{S}^{i-p-1})\mathcal{S}(a^{(i)}_{i+1,1}).$$

Final step: To Find relation between $u_{t_{n+1}}$ and u_{t_n}.

$$u_{t_{n+1}} = u(\mathcal{S} - \mathcal{S}^{-p})(\mathcal{S} - 1)^{-1}(u\mathcal{S}^{-1} - \mathcal{S}^p u)\mathcal{S}(a^{(p)}_{p+1,1}).$$
$$a^{(1)}_{2,1} = -\mathcal{S}^{p-1}(1 - \mathcal{S}^p)^{-1}\frac{u_{t_n}}{u}$$

Locality of symmetries I

$\Re$ is not a weakly nonlocal operator! Induction?!
Define homogeneous difference polynomials (Svinin '09):

$$\mathcal{P}^{(l,k)} = \sum_{0\leq\lambda_{l-1}\leq\cdots\leq\lambda_0\leq k} \left(\prod_{j=0}^{l-1} u_{\lambda_j+jp} \right),$$

where $k \geq 0, l \geq 1$ and $p \geq 1$ are all integers.

Example. For fixed p, we have

$$\mathcal{P}^{(1,k)} = \sum_{j=0}^{k} u_j \quad \text{and} \quad \mathcal{P}^{(l,0)} = u u_p u_{2p} \cdots u_{(l-1)p} \ .$$

The Narita-Itoh-Bogoyavlensky lattice

$$u_t = u(\mathcal{S} - \mathcal{S}^{-p})\mathcal{P}^{(1,p-1)}.$$

Properties of $\mathcal{P}^{(l,k)}$:

$$\mathcal{P}^{(l,k)} - \mathcal{P}^{(l,k-1)} = u_k \mathcal{S}^p(\mathcal{P}^{(l-1,k)});$$
$$\mathcal{P}^{(l,k)} - \mathcal{S}(\mathcal{P}^{(l,k-1)}) = u_{(l-1)p}\mathcal{P}^{(l-1,k)}.$$
$$\Longrightarrow (\mathcal{S}-1)\mathcal{P}^{(l,k)} = u_{k+1}\mathcal{S}^{p+1}(\mathcal{P}^{(l-1,k)}) - u_{(l-1)p}\mathcal{P}^{(l-1,k)}$$

$$(\mathcal{S}^{p-i}u - u\mathcal{S}^{-i})\mathcal{S}^{-lp+i}\mathcal{P}^{(l,(l+1)p-i)} =$$
$$= (\mathcal{S}^{p-i}u - u\mathcal{S}^{-(i+1)})\mathcal{S}^{-lp+i+1}\mathcal{P}^{(l,(l+1)p-i-1)}, \quad 0 \le i \le p.$$

Thm. $\Re^l(u_t) = u(1-\mathcal{S}^{-(p+1)})\mathcal{S}^{1-lp}\mathcal{P}^{(l+1,(l+1)p-1)}$ for all $0 \le l \in \mathbb{Z}$.

Proof. $(u - u\mathcal{S}^{-p})\mathcal{S}^{-lp+p}\mathcal{P}^{(l,lp)} = \Re^{l-1}(u_t)$.

$$\Re = u(\mathcal{S}-\mathcal{S}^{-p})(\mathcal{S}-1)^{-1}(\mathcal{S}^p u - u\mathcal{S}^{-1}) \cdot \prod_{i=1}^{\rightarrow p-1}(\mathcal{S}^{p-i}u - u\mathcal{S}^{-i})^{-1}(\mathcal{S}^{p-i}u - u\mathcal{S}^{-(i+1)}) \cdot (u - u\mathcal{S}^{-p})^{-1}$$

Miura transformations

$$D_u = u\left(\sum_{k=0}^{p-1}\frac{1}{v_k}\mathcal{S}^k\right) = u\left(\sum_{k=0}^{p-1}\mathcal{S}^k\right)\frac{1}{v}$$
$$= u\left(\sum_{k=0}^{p}\frac{1}{w_k}\mathcal{S}^k\right) = u\left(\sum_{k=0}^{p}\mathcal{S}^k\right)\frac{1}{w}.$$

$$\mathcal{H}_v = v(\mathcal{S}-1)\mathcal{S}^{-1}(\mathcal{S}^{p+1}-1)(\mathcal{S}^p-1)^{-1}v$$

$$\mathcal{H}_w = w(\mathcal{S}-1)(\mathcal{S}^p-1)(\mathcal{S}^{p+1}-1)^{-1}w$$

$$v_t = \mathcal{H}_v\delta_v\prod_{k=0}^{p-1}v_k = \Re_v\mathcal{H}_v\delta_v\frac{p\ln v}{p+1}$$

$$w_t = \mathcal{H}_w\delta_w\prod_{k=0}^{p}w_k = \Re_w\mathcal{H}_w\delta_w\ln w$$

How about the Hamiltonian structure of dSK?

Still open!