Numerical Studies of Collapses in Turbulence in Nonlinear Schrödinger Equation

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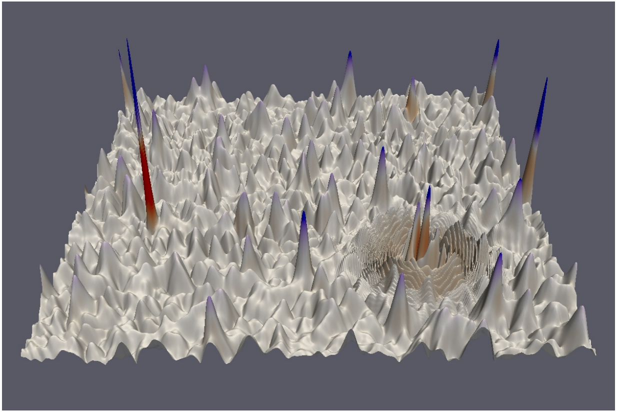
Nonlinear Schrödinger equation (NLS)



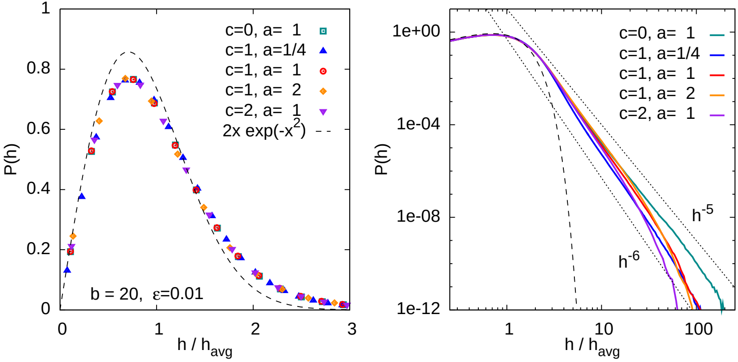
$$i\psi_t + \nabla^2\psi + |\psi|^2\psi = 0$$

$$i\psi_t + (1 - i\epsilon a)\nabla^2\psi + (1 + i\epsilon c)|\psi|^2\psi = i\epsilon b\psi$$

application: propagation of light through nonlinear media;
$\psi(x, y, t)$ — envelope of electric field

Distribution of $|\psi|$ in the field

Notations: $h \equiv |\psi|, \quad h_{avg} = \langle|\psi|\rangle \propto \sqrt{N}, \quad N = \int |\psi|^2 d^2 r.$



Turbulent background:

- pdf scales with h_{avg}
- universal shape for $h \sim h_{avg}$

Tails:

- power-law (?) for $h \gg h_{avg}$
- depend on a, c, not on b

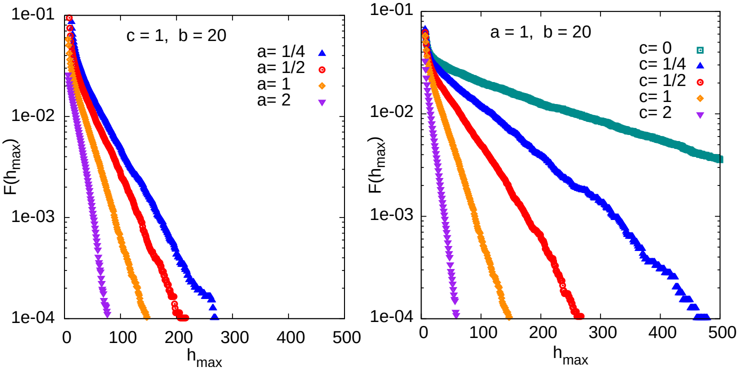
Other field quantities scale with h_{avg}

Spatial correlation length ($\sim h_{avg}$) $\leftarrow$
Correlation time ($\sim h_{avg}^2$) $\leftarrow$ depend on a, c, not on b
Spectra of $|\psi|_k^2$ ($\sim h_{avg}$) $\leftarrow$

When rescaled to h_{avg},

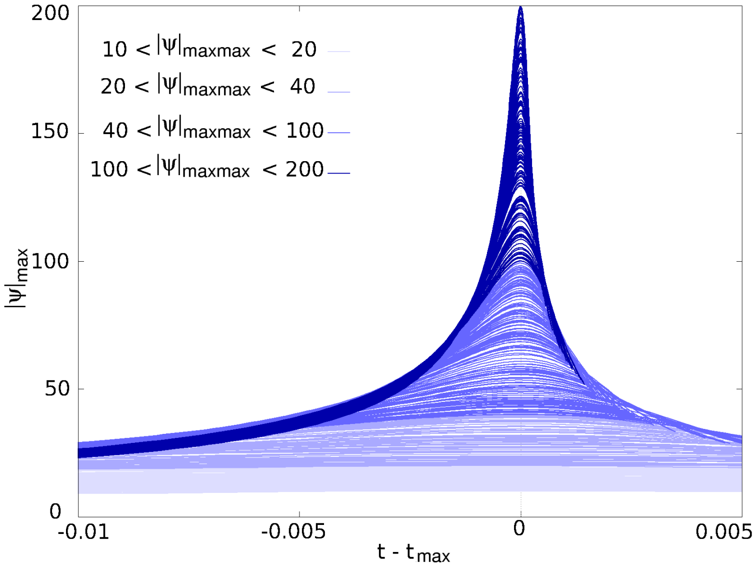
The shape of spatial correlation function is universal.
The shapes of temporal correlation function and spectra (slightly) depend on a, c.

Frequency of collapses with $h > h_{max}$.



- Frequency of collapses with $h > h_{max}$ per unit area: $F(h_{max})$.
- The dependence is exponential.
- The exponent depends on a, c, and b.
- No obvious scaling with h_{avg}.

Universality of collapses in turbulence: a=1, c=0



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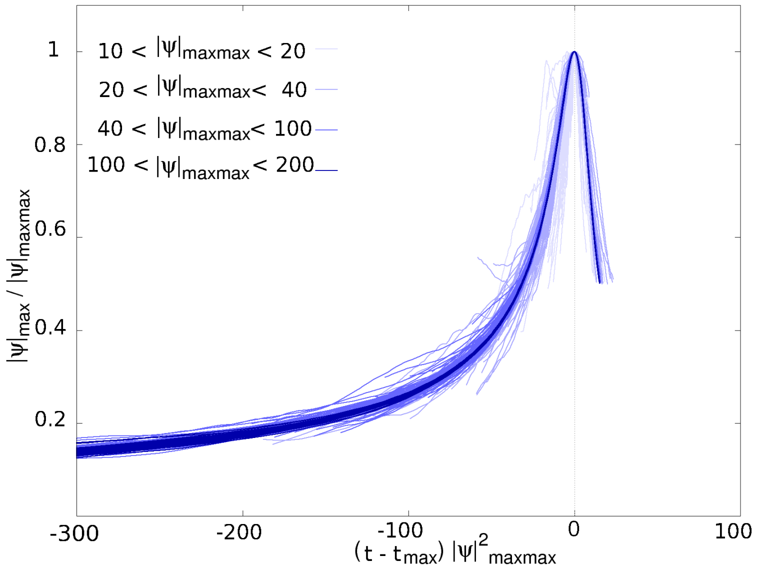

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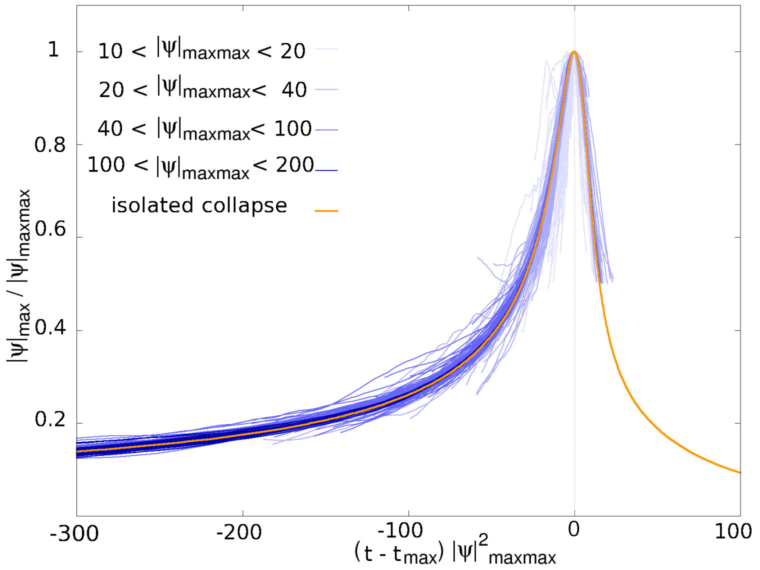

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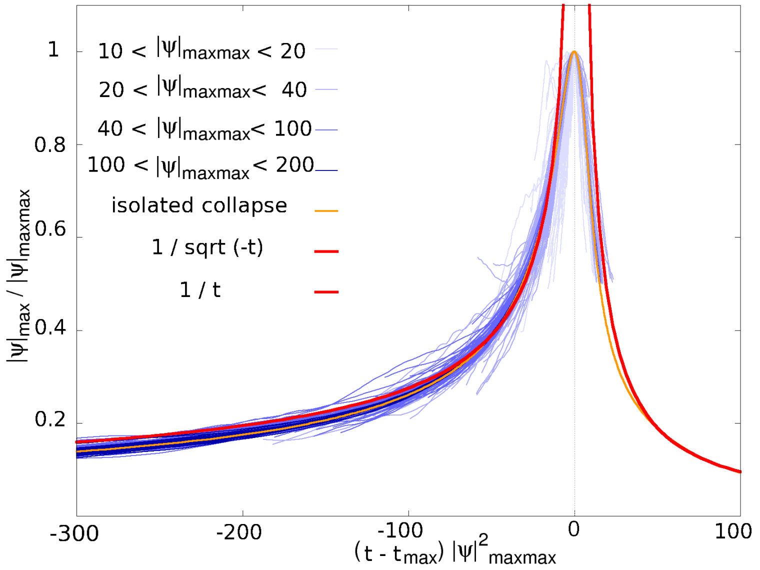
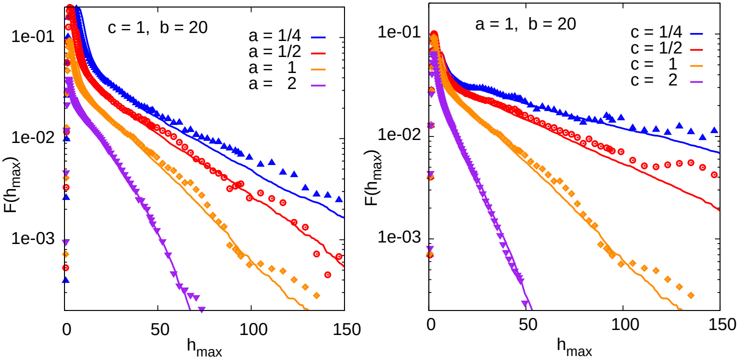


Connecting PDF of $|\psi|$ to frequency of collapses



- assume similarity among collapses, $h \sim (t_c - t)^{-\frac{1}{2}}$
- assume known frequency of collapses above $h_{\max}$: $F_{\max}(h_{\max})$
- conclude that PDF in the field $P(h) \sim h^{-5} F_{\max}(h)$
- Solid lines: $F(h_{\max})$. Points: $0.012\, h^5 P(h)$.

Diagnostics of collapses

- In slow variables $\rho = \frac{r}{L}$ and $\tau = \int L^{-2}(t')dt'$

$$\psi(r,t) = \frac{1}{L} V(\rho,\tau) e^{i\tau + i\gamma(\tau)\rho^2/4} \qquad (1)$$

  critical NLS becomes

$$\nabla^2 V + V^3 - V + \frac{1}{4}\beta\rho^2 V = 0. \qquad (2)$$

- Solve (2) for each β, tabulate $V_0(\beta)$ at the center.
- Taylor expansion of (2) at the center and (1) give

$$L = L\,(\,|\psi|(0)|,\ |\psi|''_{rr}(0)\,) \ \Rightarrow\ V_0 \ \Rightarrow\ \beta; \qquad \gamma = 2L^2\phi''_{rr}(0)$$

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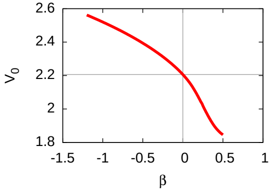

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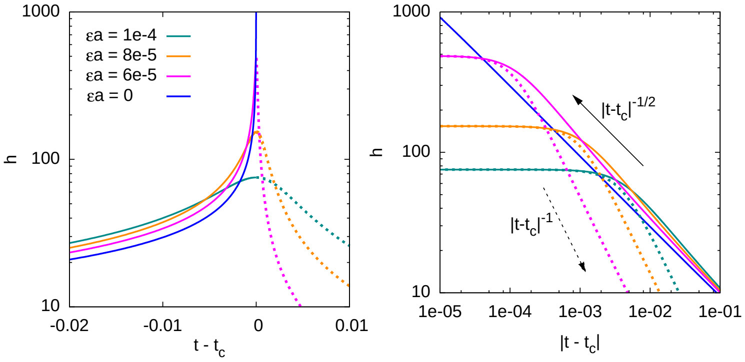
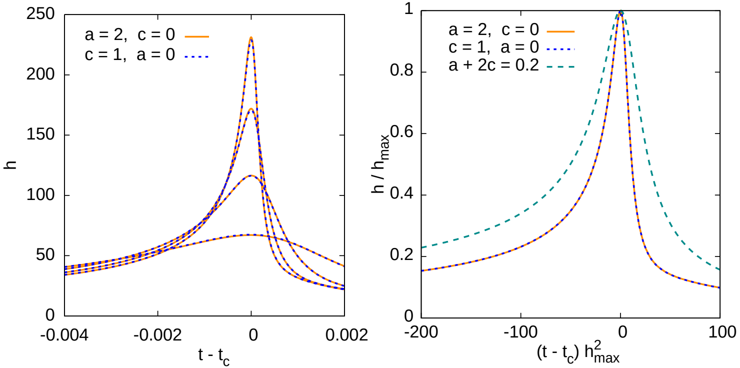
Collapse growth and decay



Evolution of collapses with $h_0 = 2.8$, $r_0 = 1$ with $c = 0$, $b = 0$.
Growth rate depends on a and c, decay rate does not.
Notation: $h \equiv |\psi|_{r=0}$.

Rescaled evolution: $(a + 2c)$ similarity



Left:

NLS coordinates

collapses with $\epsilon = 0.01$, $a + 2c = 2$, and different h_0.

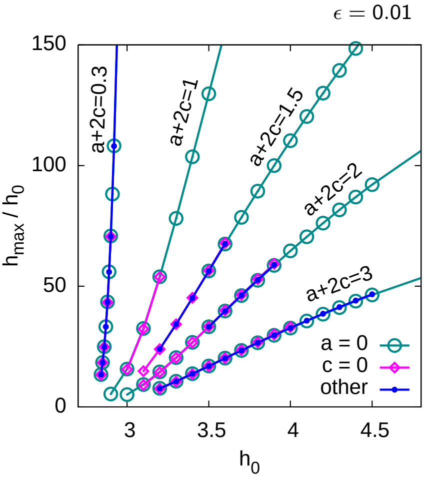
Right:

coordinated rescaled with $h_{\max}$;

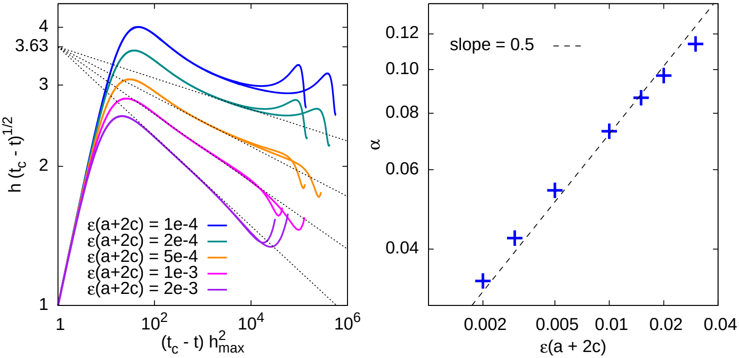
same curve for collapses with $a + 2c = 2$,

different for $a + 2c = 0.2$

Maximum height of collapses: $(a + 2c)$ similarity



Growth rate of collapses: $(a + 2c)$ similarity



$$h \propto (t_c - t)^{-\left(\frac{1}{2}+\alpha\right)}, \qquad \alpha = \alpha(a + 2c)$$

Left: $h(t)$, compensated by $(t_c - t)^{-\frac{1}{2}}$.
Right: coefficient α determined from dotted lines on the left panel.

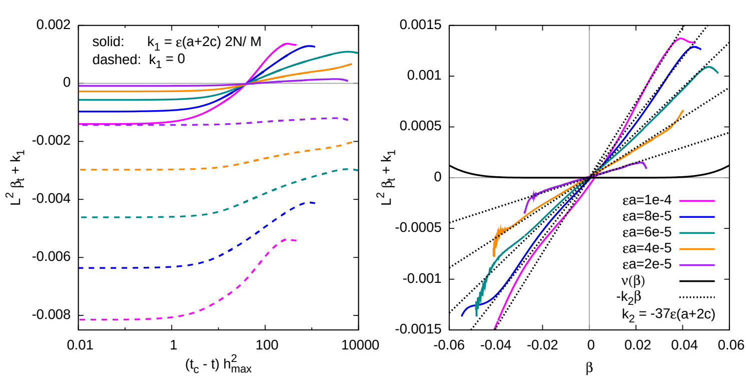
ODE model for collapse growth (Fibich and Levy, 1998)

$$
\begin{aligned}
L^2\beta_t &= -k_1 - k_2\beta + k_3(LL_t)^2 - \nu(\beta) \\
L^3 L_{tt} &= -\beta
\end{aligned}
$$

$$
\nu(\beta) = 45.1e^{-\pi/\sqrt{|\beta|}}
$$

$$
\begin{aligned}
k_1 &= 2\epsilon(a+2c)\frac{N}{M} \\
k_2 &= \epsilon a\frac{P-N}{M} \\
k_3 &= \epsilon a\frac{P-N-2M}{M}
\end{aligned}
\qquad
\begin{aligned}
N &= 2\pi\int \rho R^2 d\rho \approx 11.69 \\
P &= 2\pi\int \rho^3 R_\rho^2 d\rho \approx 18.65 \\
M &= \frac{\pi}{2}\int \rho^3 R^2 d\rho \approx 3.46
\end{aligned}
$$

Test of ODE model: term-by-term comparison



$$
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L^2\beta_t &= -k_1 - k_2\beta + k_3(LL_t)^2 - \nu(\beta) \\
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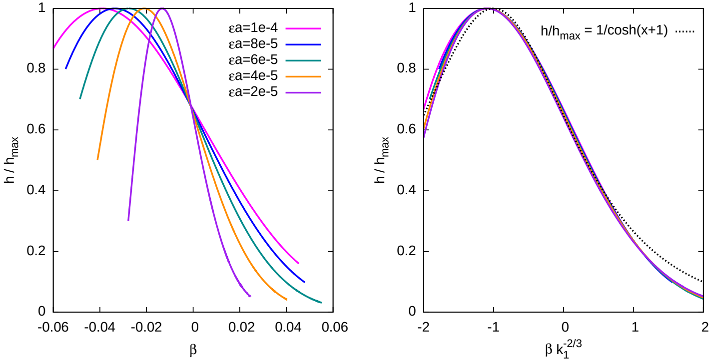
\end{aligned}
$$

Data suggest: k_1 term describes first order effects well;
$k_2 \approx -37\epsilon(a+2c)$ instead of original $k_2 \approx 2\epsilon a$.

Side note on ODE model



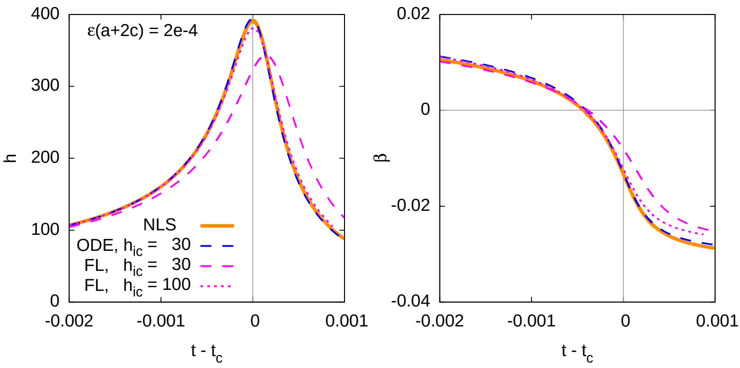
In slow variables keep first two terms of ODE model:

$$\beta_\tau = -k_1 - k_2\beta$$
$$L_{\tau\tau} - 2L_\tau^2/L = -\beta L$$

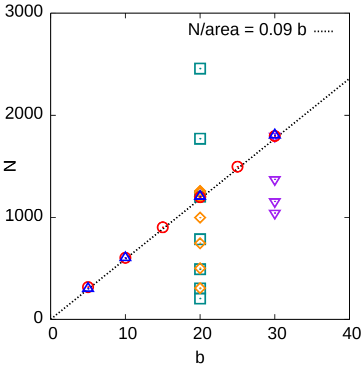
Results in $\frac{h}{h_{\max}} = \frac{L_{\max}}{L} \approx \cosh^{-1}\left[-\left(\tilde\beta - \tilde\beta_{\max}\right)|\tilde\beta_{\max}|^{\frac{1}{2}}\right],$

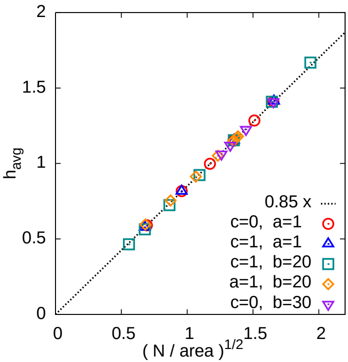
where $\tilde\beta = \beta k_1^{-\frac{2}{3}}$ and $\tilde\beta_{\max} = -0.92$

Test of ODE model: collapse evolution



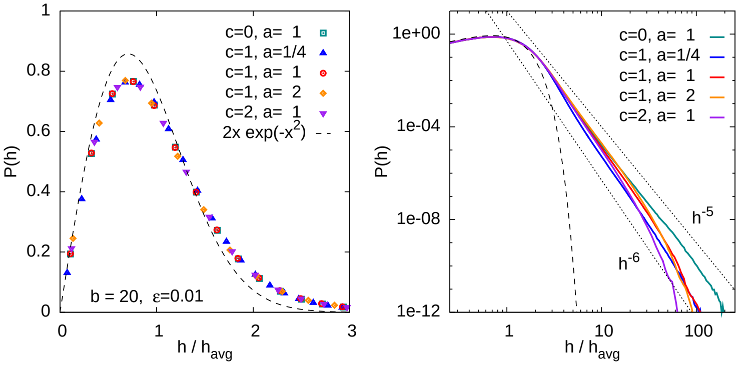
Parameter space: a, c, and b ($\epsilon = 0.01$).





- Notations: $h \equiv |\psi|$, $h_{avg} = \langle|\psi|\rangle$, $N = \int |\psi|^2 d^2 r$.
- No obvious scalings with a, c, or b...
  ...except $N \propto b$ and $h_{avg} \propto \sqrt{N}$.
- We will show that h_{avg} well describes background turbulence.
- What determines h_{avg}?

Distribution of $|\psi|$ in the field scales with h_{avg}



- PDF of $|\psi|$ scales with h_{avg}.
- Universal shape for $h \sim h_{avg}$, power-law (?) tails for $h \gg h_{avg}$.
- The tails do not depend on b.
- The tails do depend on a and c.

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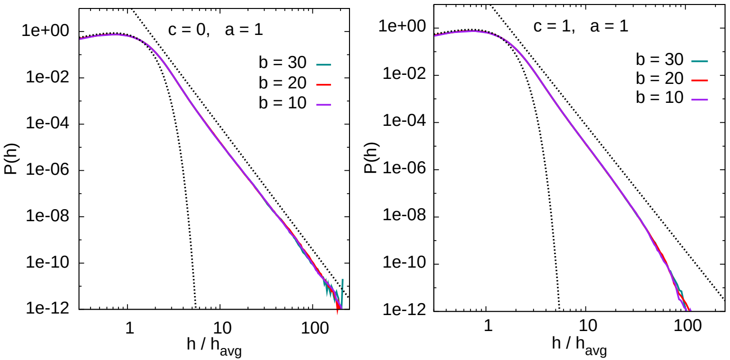

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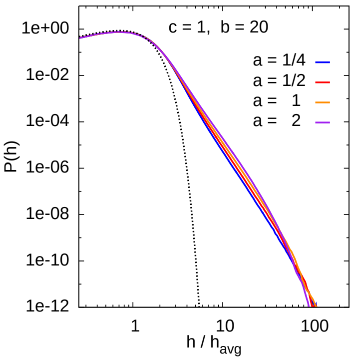

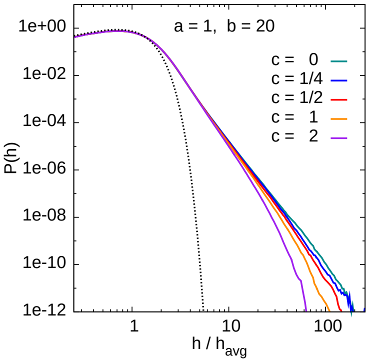

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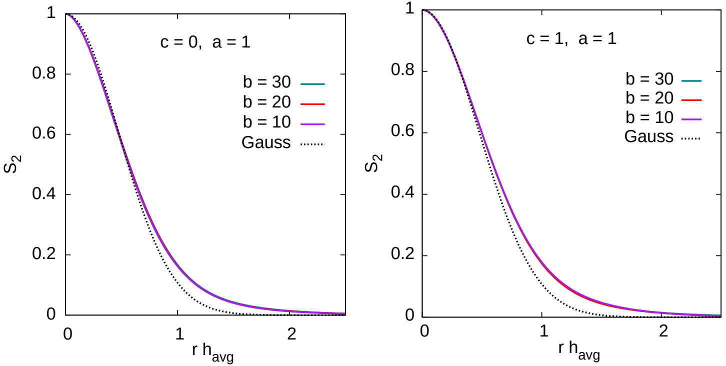
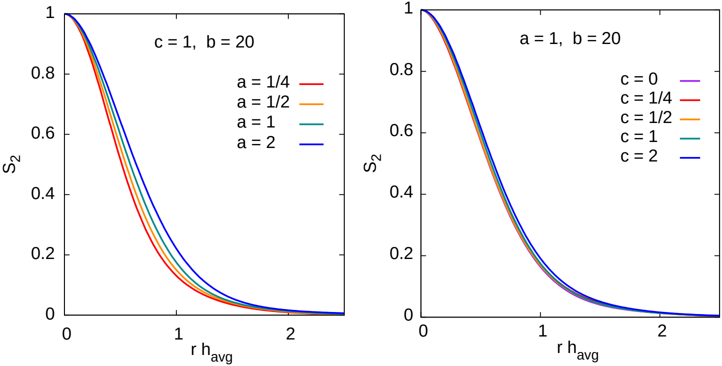
Spatial correlation function scales with h_{avg}



- Spatial correlation function scales with h_{avg}.
- Correlation length does not depend on b.
- Correlation length depend on a and c.
- The shape, when rescaled to correlation length, is universal.

Spatial correlation function scales with h_{avg}



- Spatial correlation function scales with h_{avg}.
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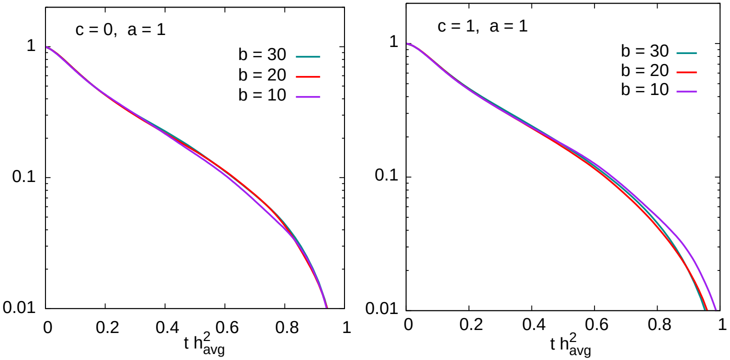
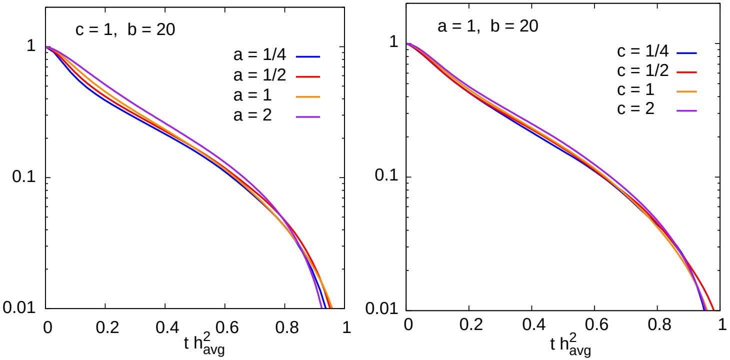
Temporal correlation function scales with h^2_{avg}



- Temporal correlation function scales with h^2_{avg}.
- Correlation time does not depend on b.
- Correlation time depend on a and c.

Temporal correlation function scales with h_{avg}^2



- Temporal correlation function scales with h_{avg}^2.
- Correlation time does not depend on b.
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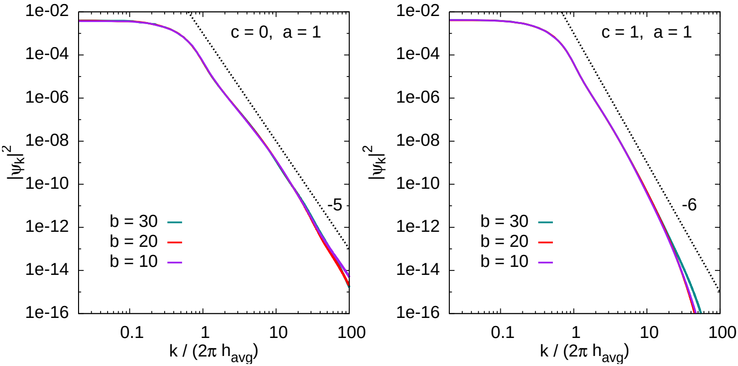
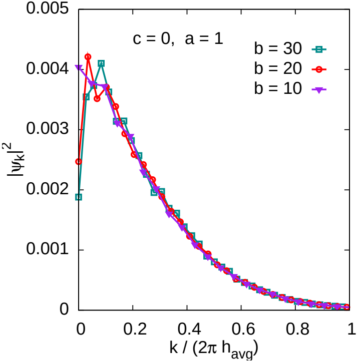
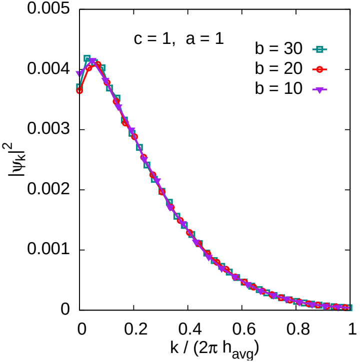
Energy spectra scale with h_{avg}



- Spectra of $|\psi|_k^2$ scale with h_{avg}.
- The tails do not depend on b.
- The tails do depend on a and c.

Energy spectra scale with h_{avg}





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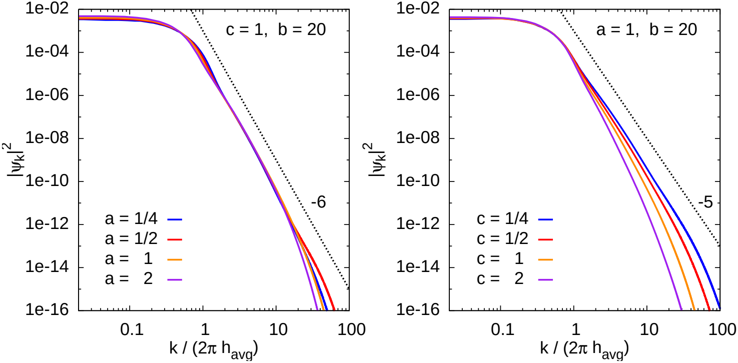

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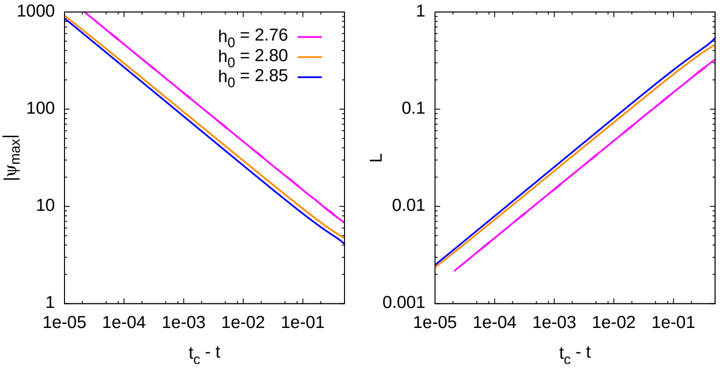
Test: critcal collapses



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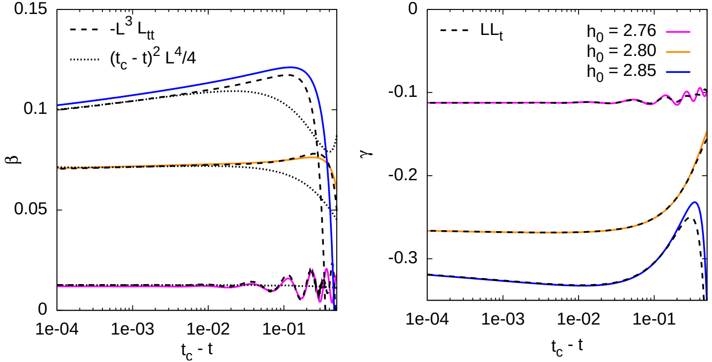

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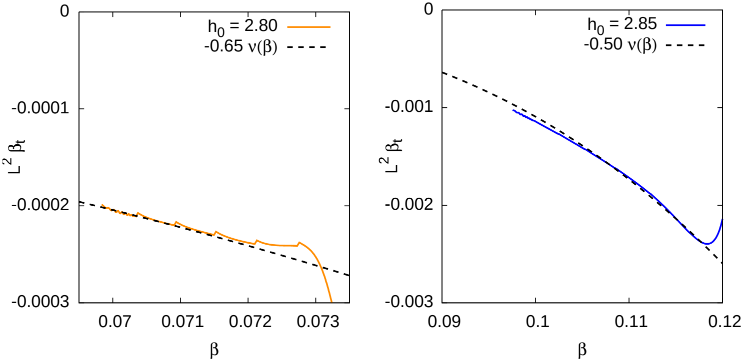
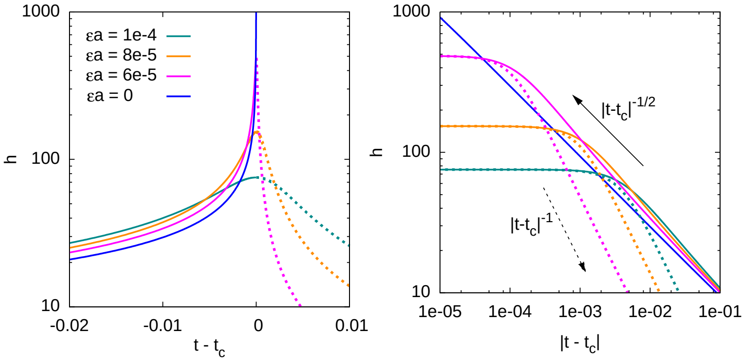


Beta and gamma with stabilization



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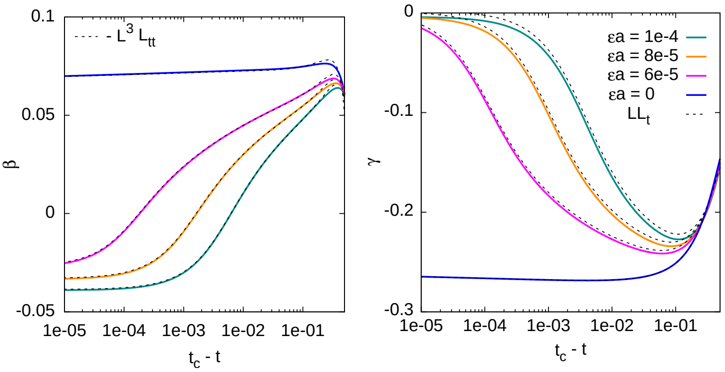

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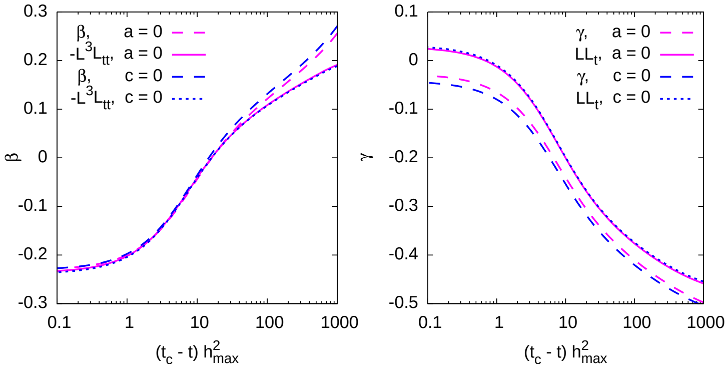

Here, $\epsilon(a+2c) = 0.01$. Notice that (a+2c) similarity is more prononeced in L and its derivatives rather than in β and γ.