On Nonlinear stage of the Modulational Instability

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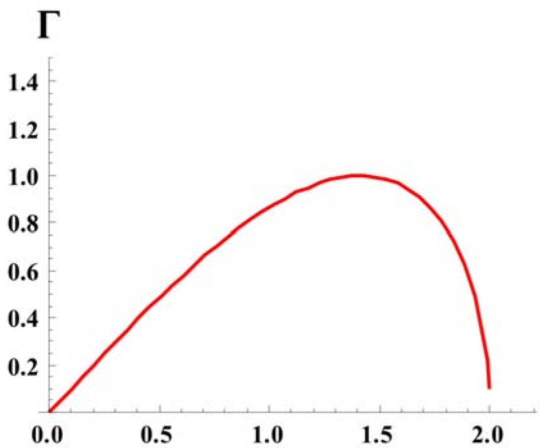
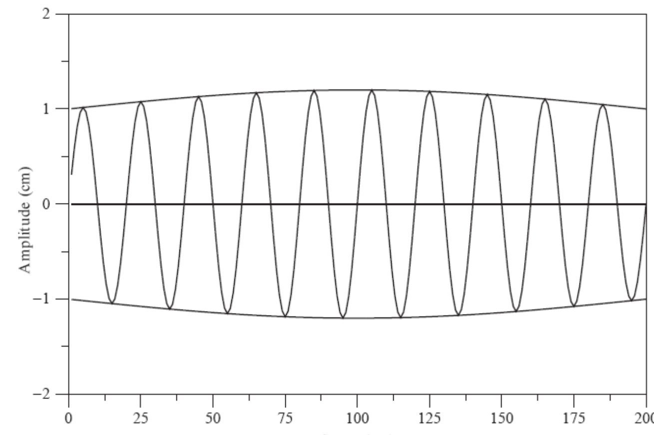
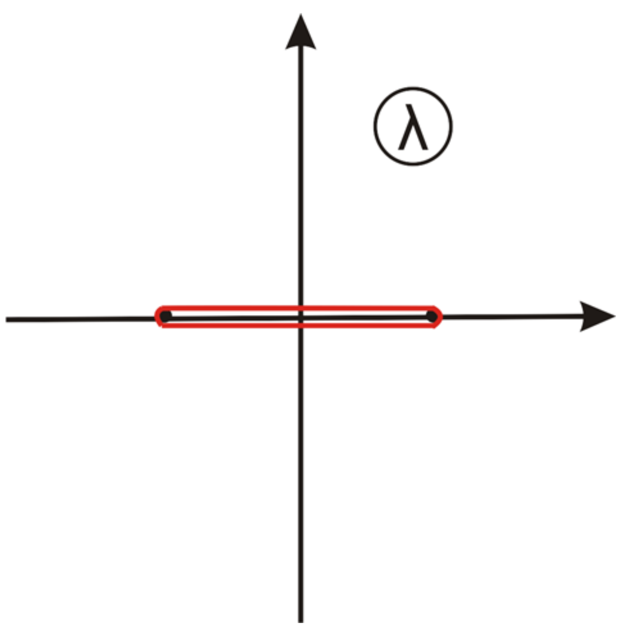
Introduction



Modulation instability -
Zakharov, Benjamin and Feir



NLSE $$i\varphi_t - \tfrac{1}{2}\varphi_{xx} - (|\varphi|^2 - A^2)\varphi = 0$$



Complete integrability –
Zakharov and Shabat, 1971

Nonlinear stage of the MI. NLSE
with periodic boundary conditions –
Kotljarov, 1976; Yuen and Lake, 1982
What about evolution of localized small initial condition?

Dressing method

NLSE $i\varphi_t - \frac{1}{2}\varphi_{xx} - (|\varphi|^2 - A^2)\varphi = 0$ is the compatibility condition for the following overdetermined linear system for a matrix function Ψ:

$$\frac{\partial \Psi}{\partial x} = \widehat{U}\Psi, \quad i\frac{\partial \Psi}{\partial t} = (\lambda \widehat{U} + \widehat{W})\Psi$$

$$\widehat{U} = I\lambda + u; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad u = \begin{pmatrix} 0 & \varphi \\ -\overline{\varphi} & 0 \end{pmatrix}; \quad \widehat{W} = \frac{1}{2}\begin{pmatrix} |\varphi|^2 - |A|^2 & \varphi_x \\ \overline{\varphi_x} & -|\varphi|^2 + |A|^2 \end{pmatrix}$$

Ψ0 correspond to the trivial solution *φ*=*A:*

$$\Psi_0 = \frac{1}{\sqrt{1-q^2}}\begin{pmatrix} e^{\phi} & qe^{-\phi} \\ qe^{\phi} & e^{-\phi} \end{pmatrix}$$

$$\phi = kx + \Omega t, \quad k^2 = \lambda^2 - A^2, \quad \Omega = -i\lambda k, \quad q = -\frac{A}{\lambda + k}$$

We introduce the function $\chi = \Psi\Psi_0^{-1}$ which satisfy the equation:

$$\frac{\partial \chi}{\partial x} = \widehat{U}\chi - \chi\widehat{U}_0 \qquad \frac{\partial \chi^{-1}}{\partial x} = -\chi^{-1}\widehat{U} + \widehat{U}_0\chi^{-1}$$

$$\chi = 1 + \frac{R}{\lambda} + \cdots \qquad \varphi = A - 2R_{12}$$

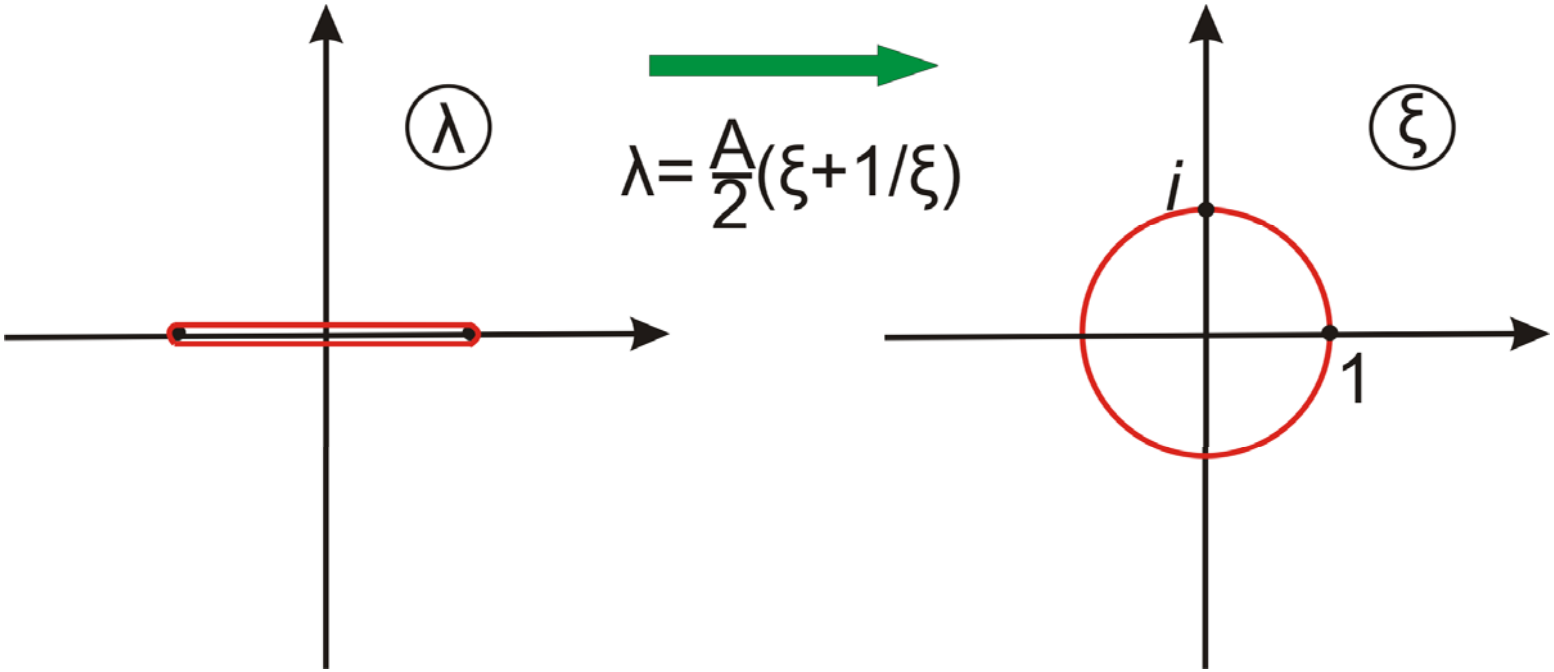
General N-solitonic solution

$$\varphi(x,t) = A - 2\widetilde{\chi_{\alpha\beta}} \qquad \widetilde{\chi_{\alpha\beta}} = \frac{\widetilde{M_{\alpha\beta}}}{M} \qquad M = \det(M_{nm})$$

$$\widetilde{M_{\alpha\beta}} = \begin{vmatrix} 0 & q_{1,\beta} & \cdots & q_{n,\beta} \\ q^*_{1,\alpha} & & & \\ \vdots & & \left(\quad M^+_{nm} \quad \right) & \\ q^*_{n,\alpha} & & & \end{vmatrix}$$

$$q^*_{n,\alpha} = \Psi_{0,\alpha\gamma}(-\lambda^*_n) n_\gamma \qquad n_\beta = \begin{pmatrix} 1 \\ C_\beta \end{pmatrix}$$

Uniformization



General One- solitonic solution

$$\varphi = A\left[\frac{\cos(2\alpha)\cosh(2u+\mu)+\left(\frac{1+R^4}{R(R^2+1)}\right)\cos(\alpha)\cos(2v-\theta)}{\cosh(2u+\mu)+\frac{2R\cos(\alpha)}{1+R^2}\cos(2v-\theta)}\right.$$
$$\left.+i\frac{\sin(2\alpha)\sinh(2u+\mu)+\left(R-\frac{1}{R}\right)\cos(\alpha)\sin(2v-\theta)}{\cosh(2u+\mu)+\frac{2R\cos(\alpha)}{1+R^2}\cos(2v-\theta)}\right]$$

Где:

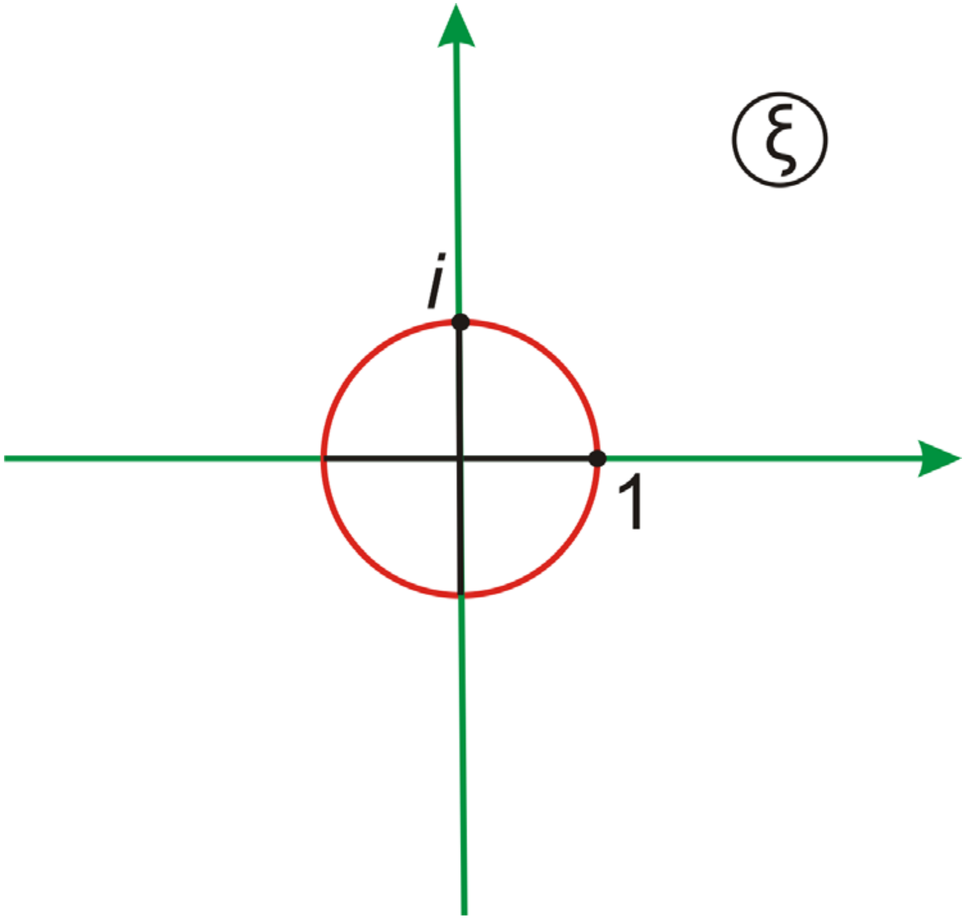
$$\phi = u + iv, \qquad u = \frac{1}{2}(æx - \gamma t), \qquad v = \frac{1}{2}(kx - \omega t)$$

$$\phi = \frac{1}{2}(æx - \gamma t) + i\frac{1}{2}(kx - \omega t)$$

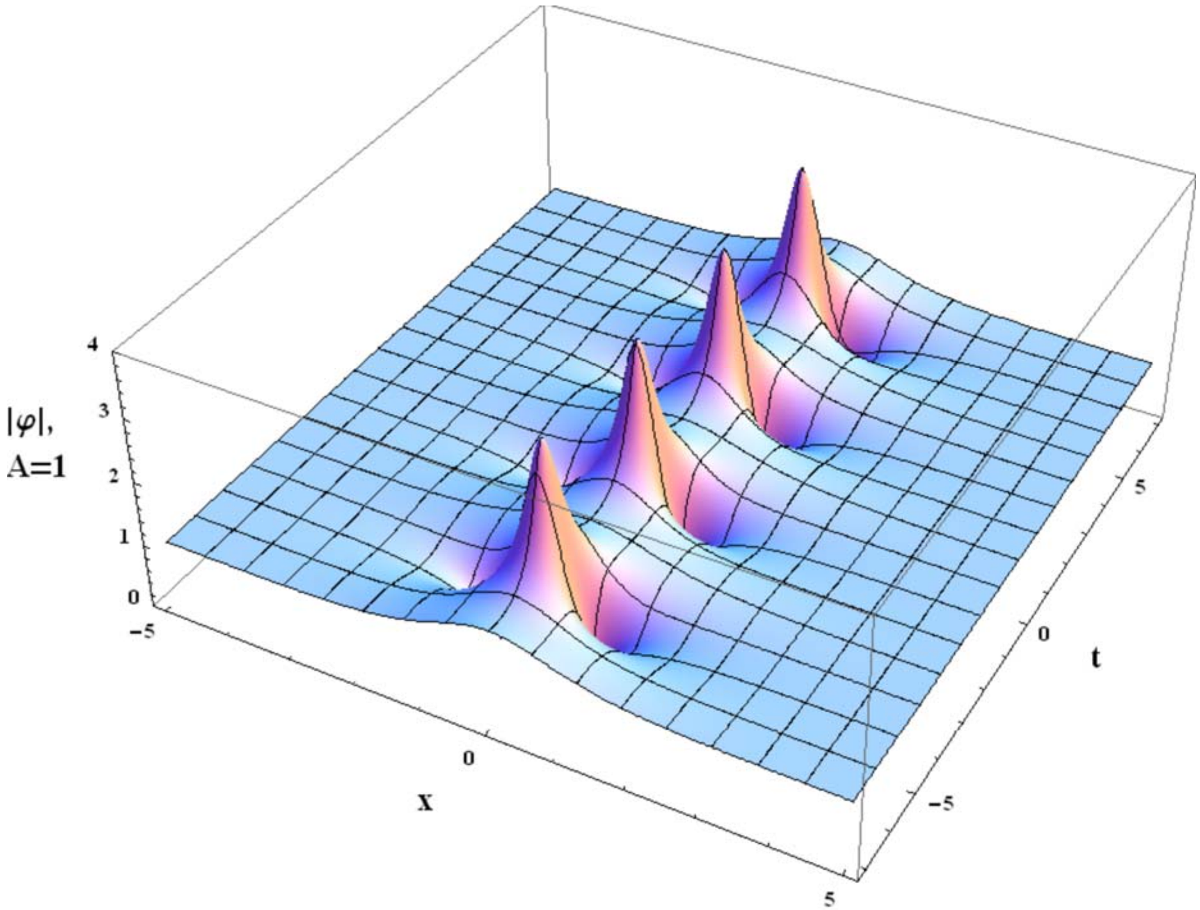
$$k = A\left(R + \frac{1}{R}\right)\sin(\alpha), \quad \omega = \frac{A^2}{2}\left(R^2 - \frac{1}{R^2}\right)\cos(2\alpha)$$

$$æ = A\left(R - \frac{1}{R}\right)\cos(\alpha), \gamma = -\frac{A^2}{2}\left(R^2 + \frac{1}{R^2}\right)\sin(2\alpha)$$

General One-solitonic solution

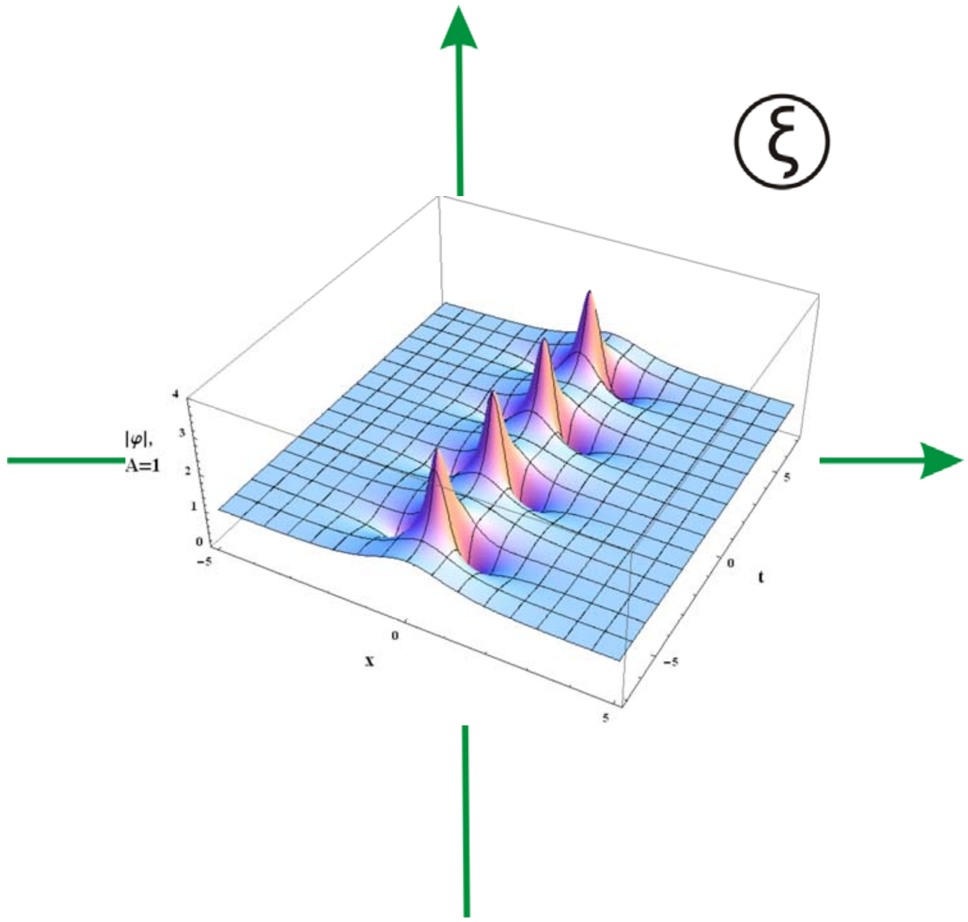
- 

General One-solitonic solution



Kuznetsov, 1976

General One-solitonic solution



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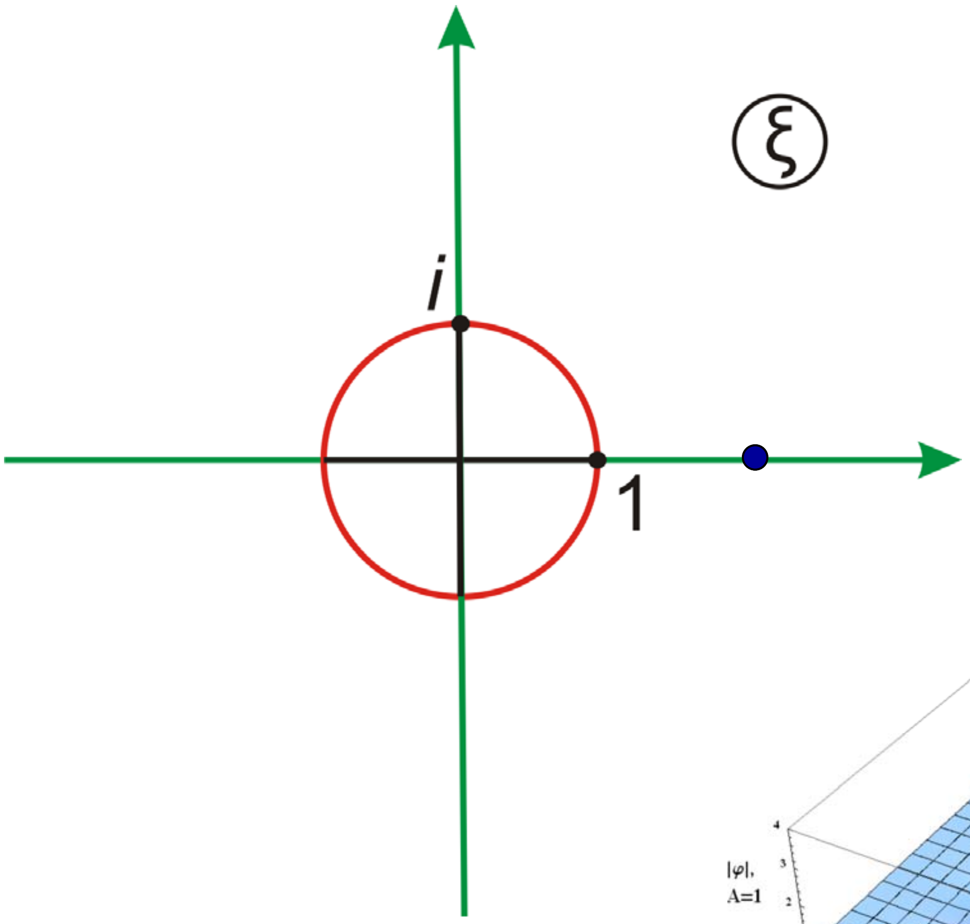
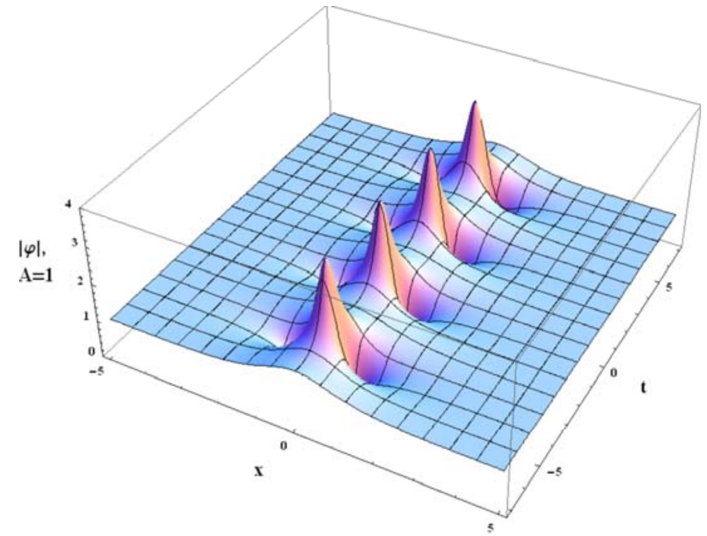
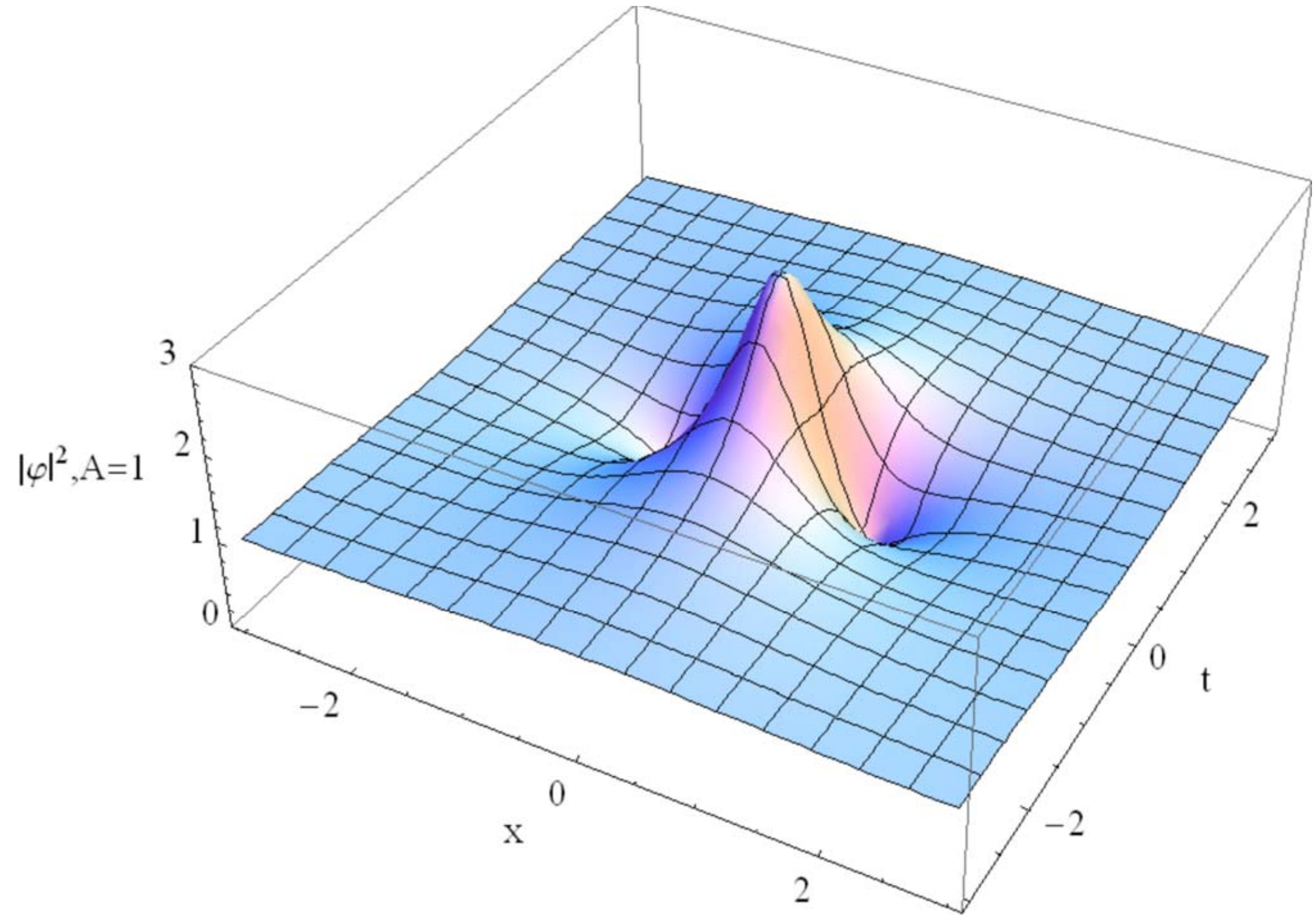




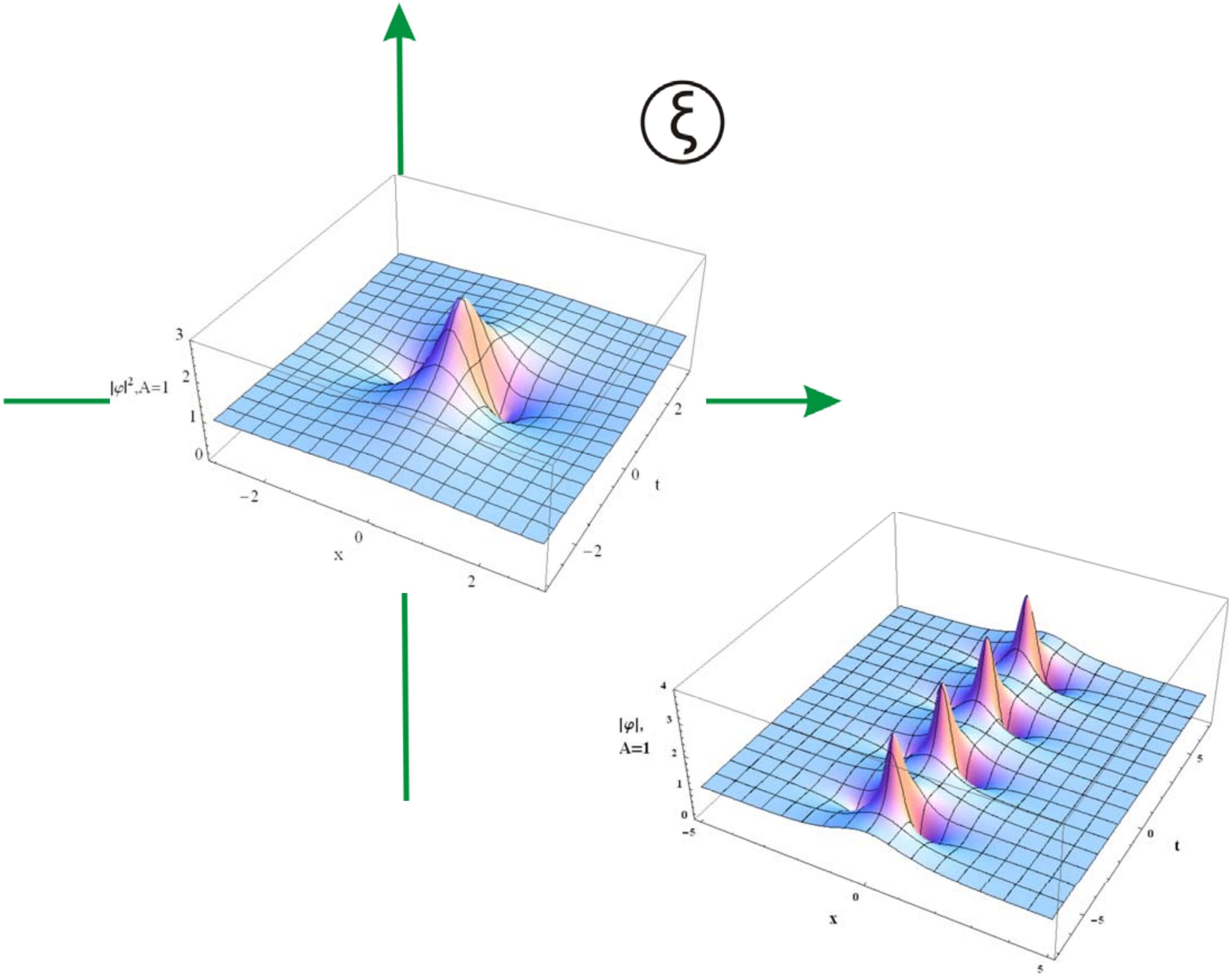
General One-solitonic solution



Peregrine, 1982

$$\varphi(x,t) = 1 - 4\frac{2it - 1}{1 + 4x^2 + 4t^2}$$

General One-solitonic solution



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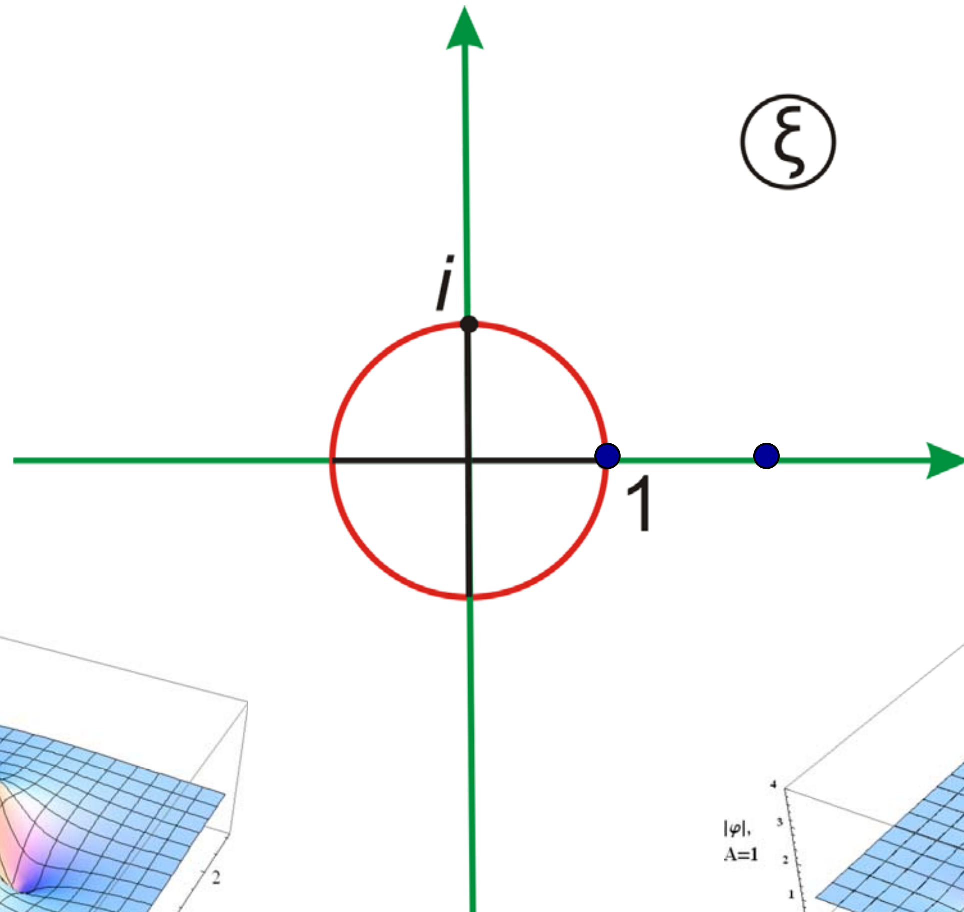

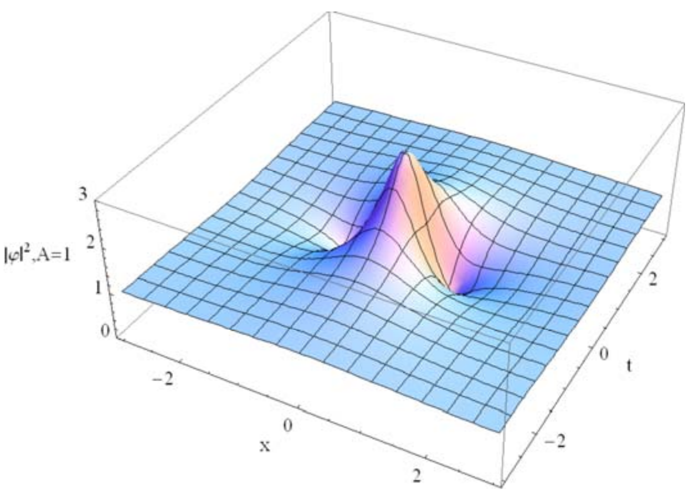

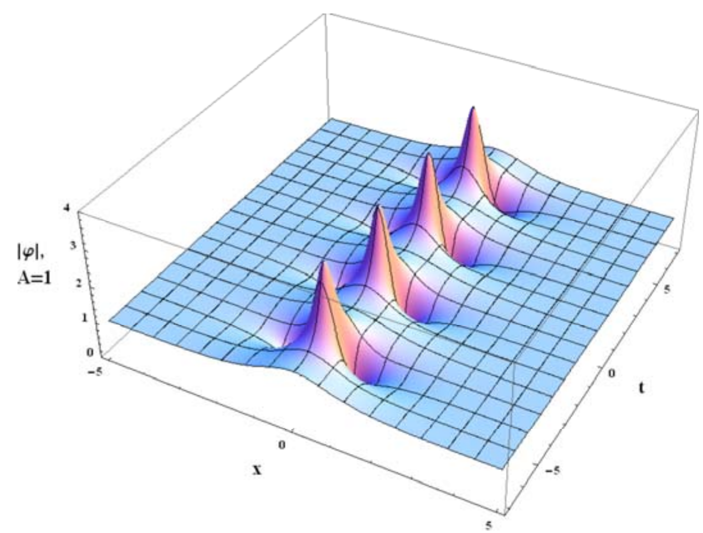

General One-solitonic solution

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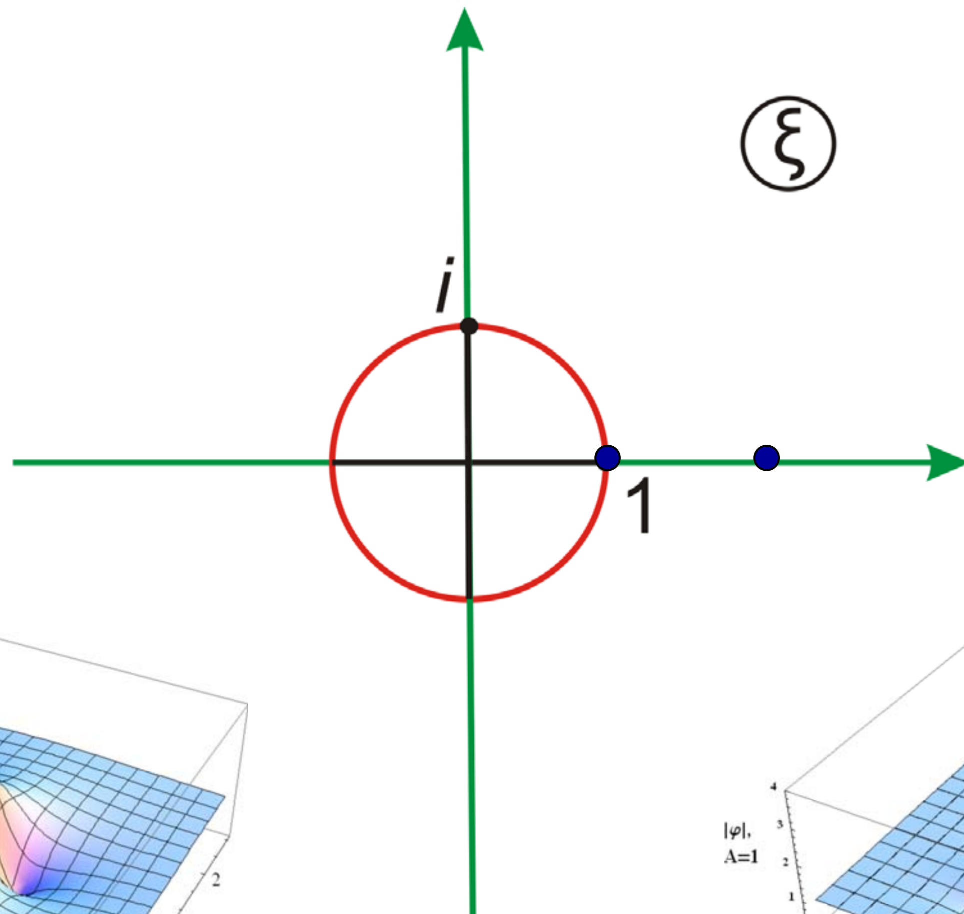

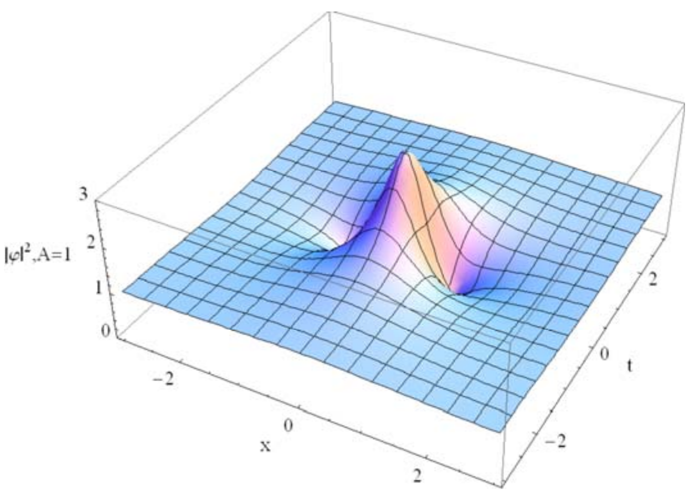

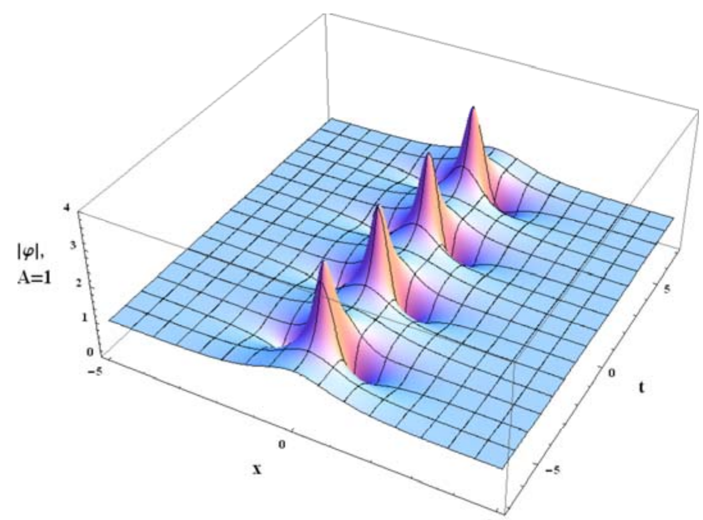

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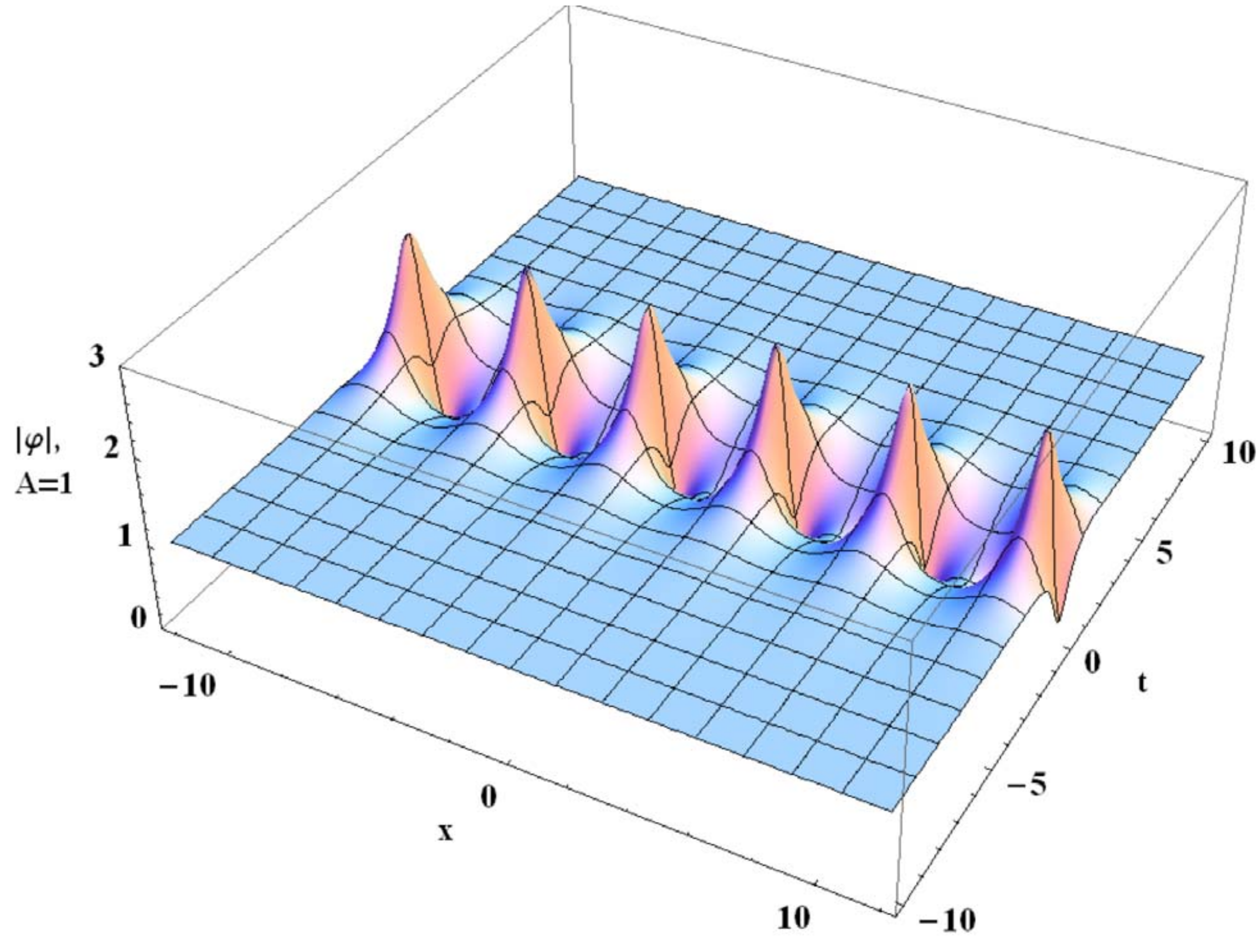
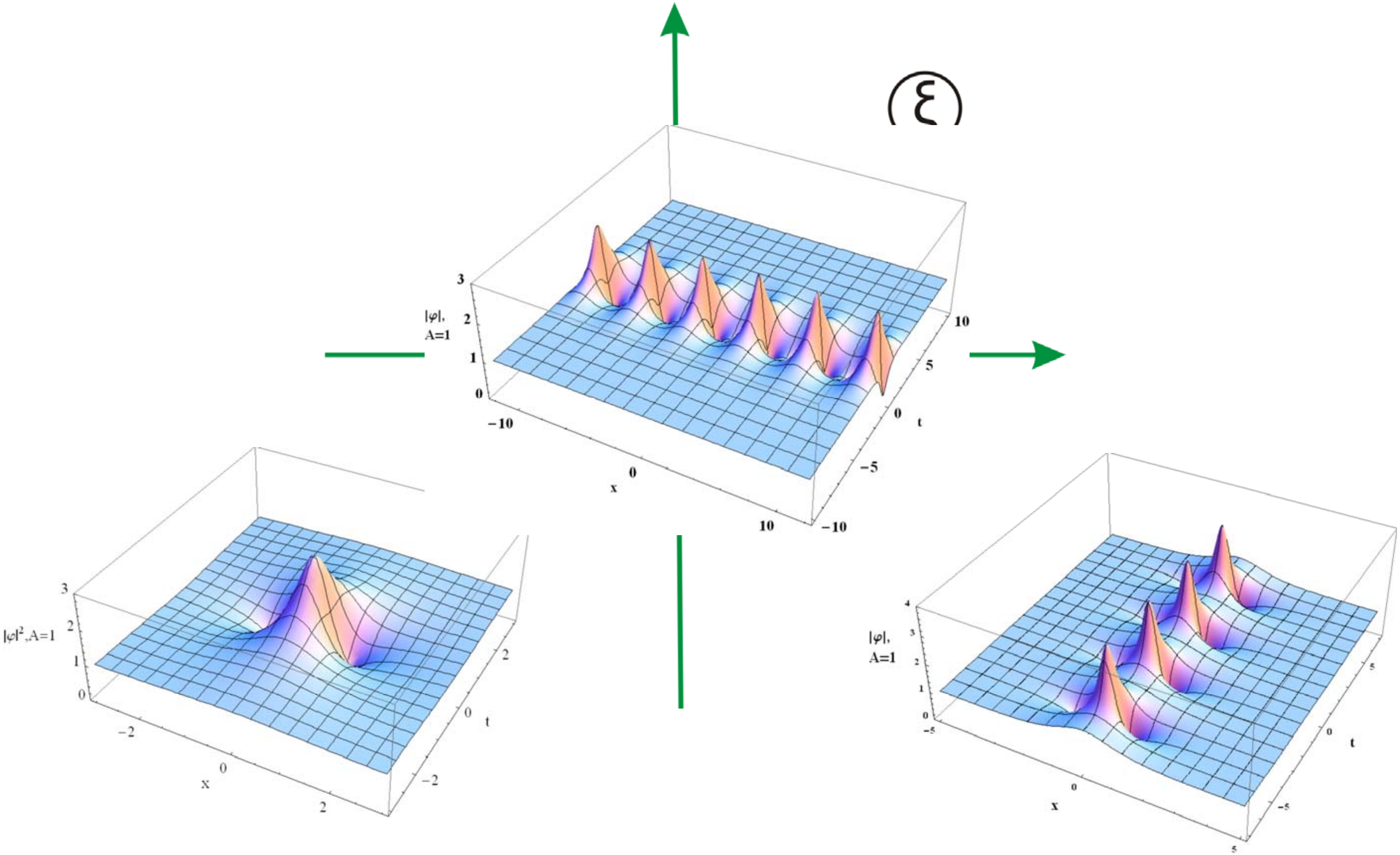


Akhmediev and Korneev, 1986

General One-solitonic solution



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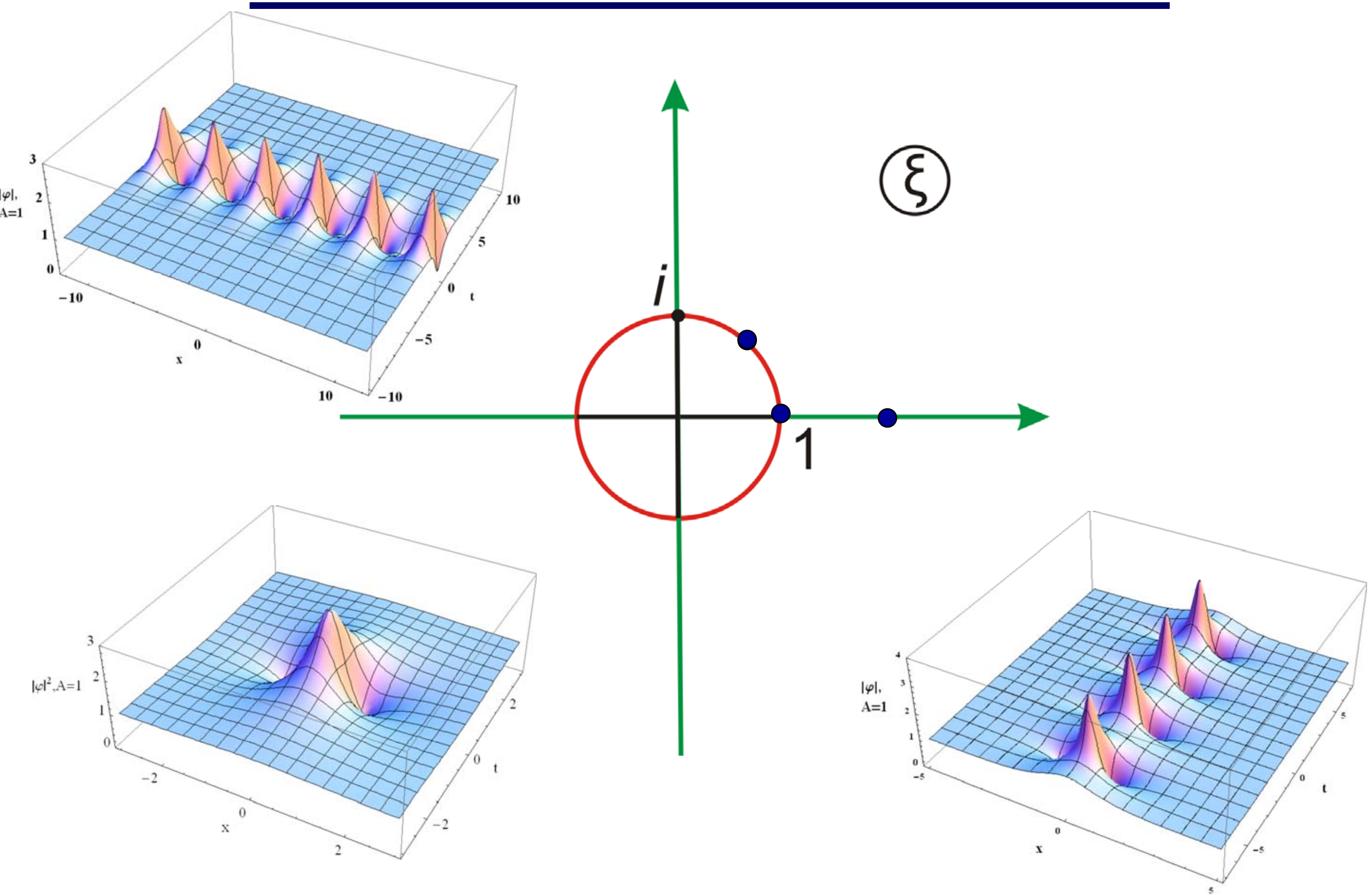

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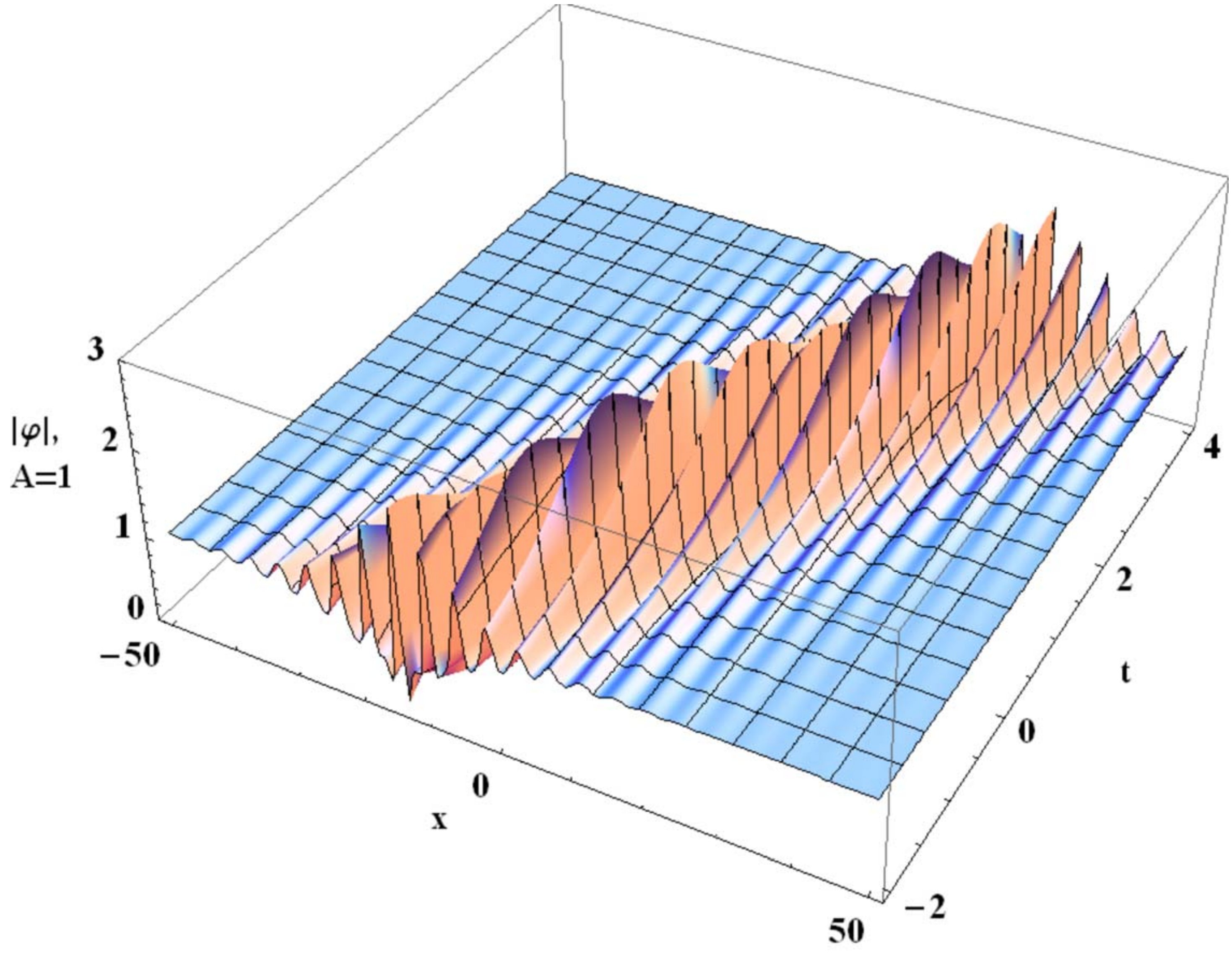

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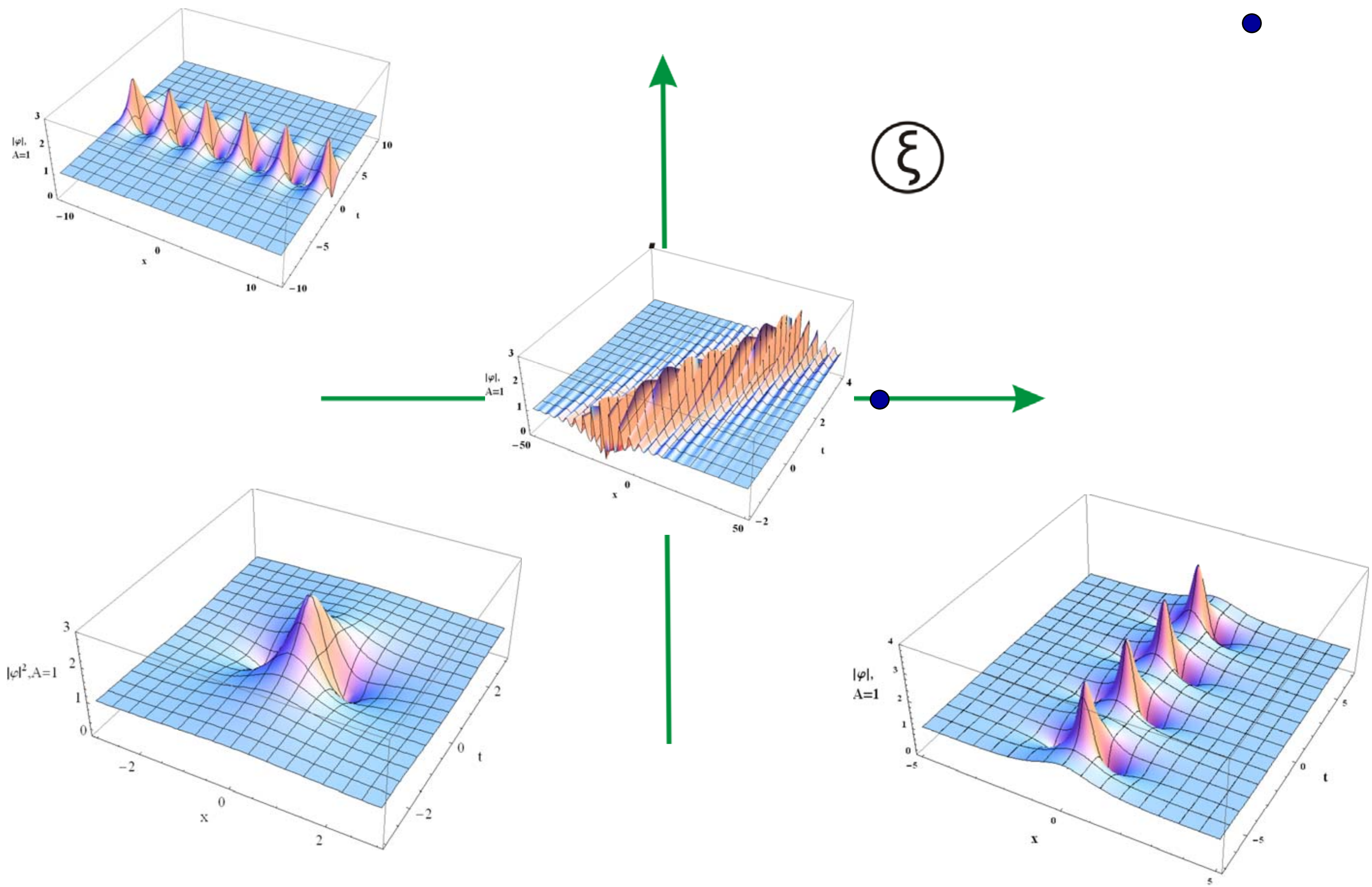

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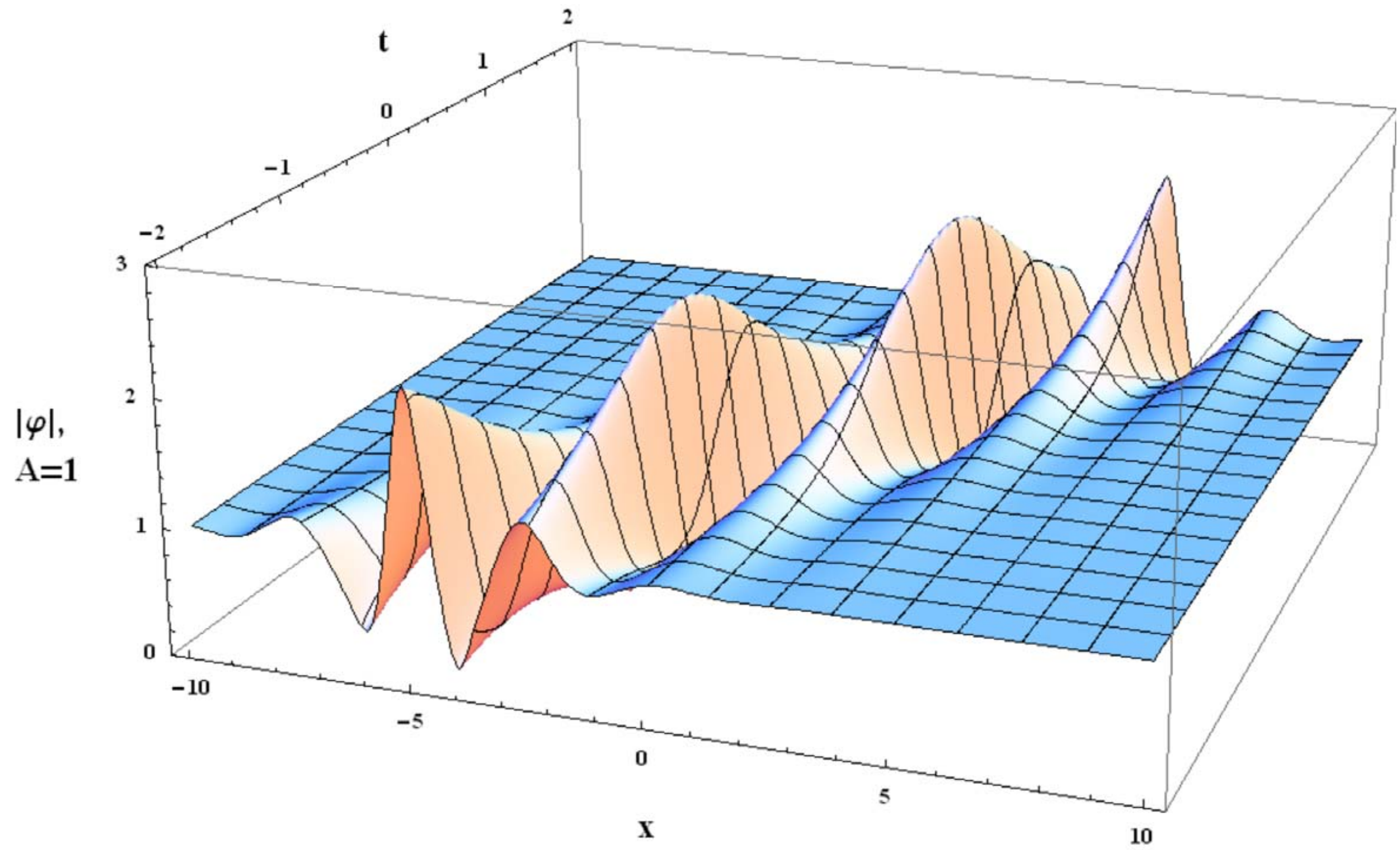
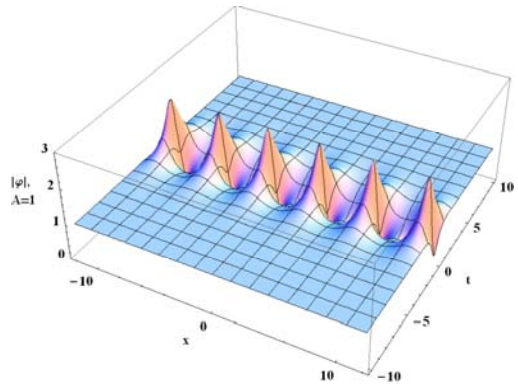
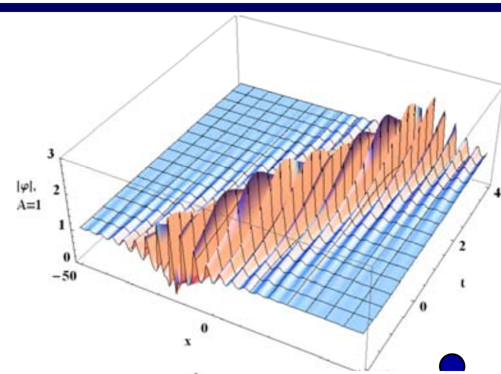


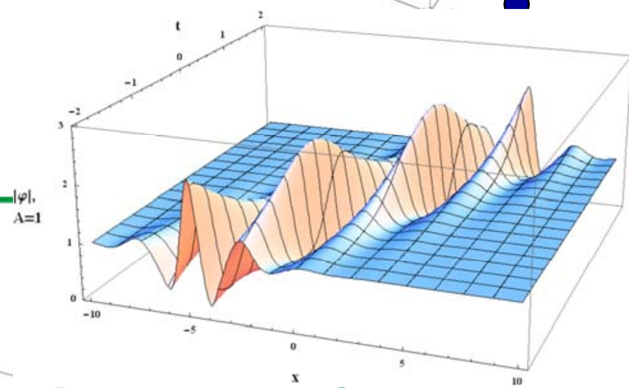
Tajiri and Watanabe; Slunyaev 1998

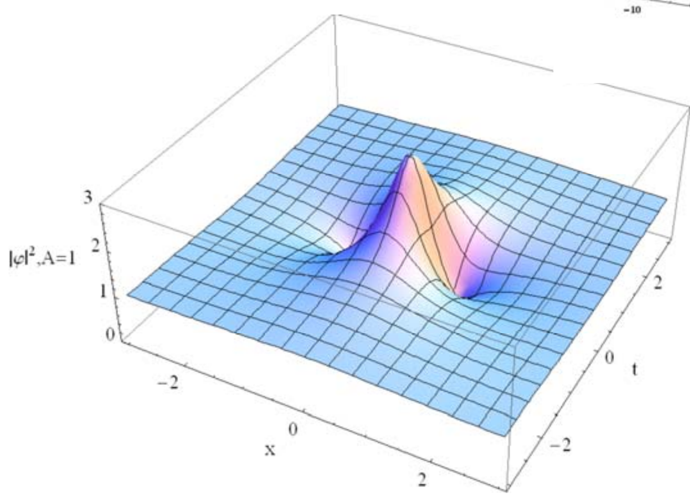
General One-solitonic solution

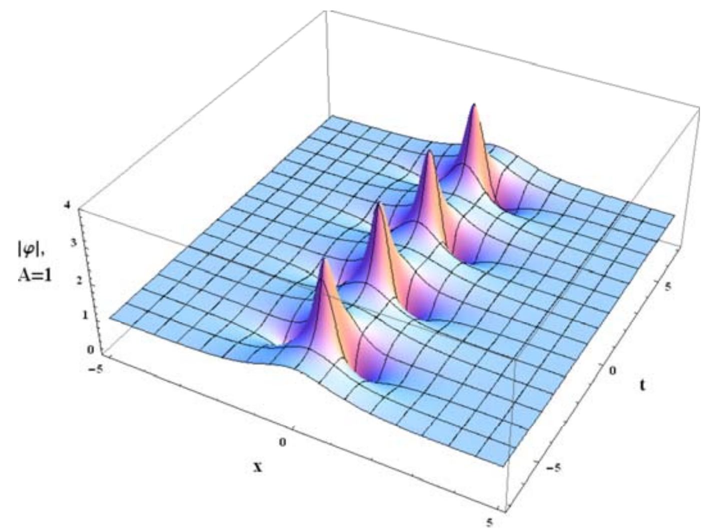




ξ







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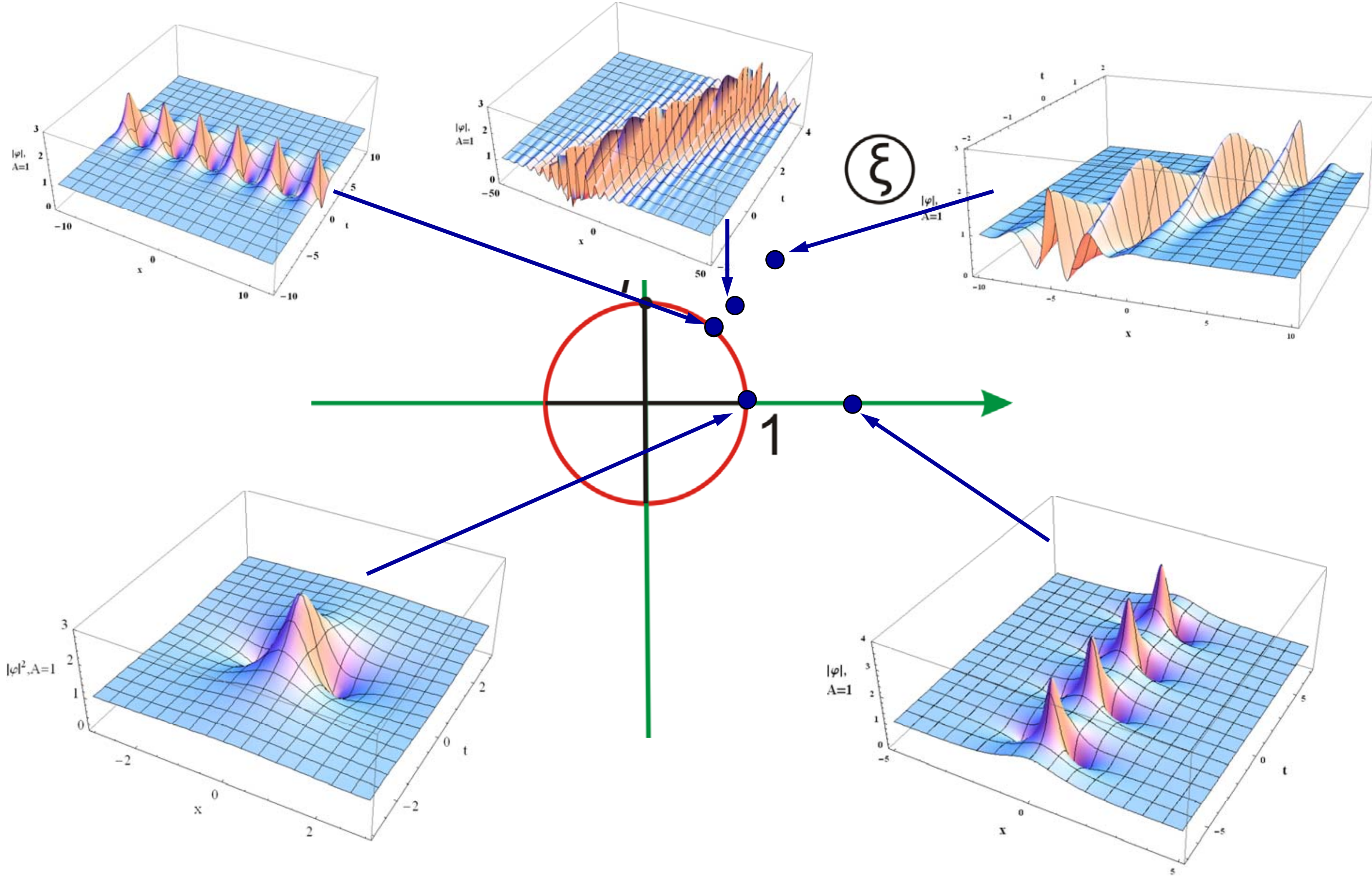


Two- solitonic solution

$$\varphi = A - \frac{2}{\Delta}\left\{ \left[\frac{|q_2|^2}{\eta+\eta^*} q_{11}^* - \frac{(q_1^* q_2)}{2\eta^*} q_{21}^*\right] q_{12} + \left[-\frac{(q_1 q_2^*)}{2\eta} q_{11}^* + \frac{|q_1|^2}{\eta+\eta^*} q_{21}^*\right] q_{22}\right\}$$

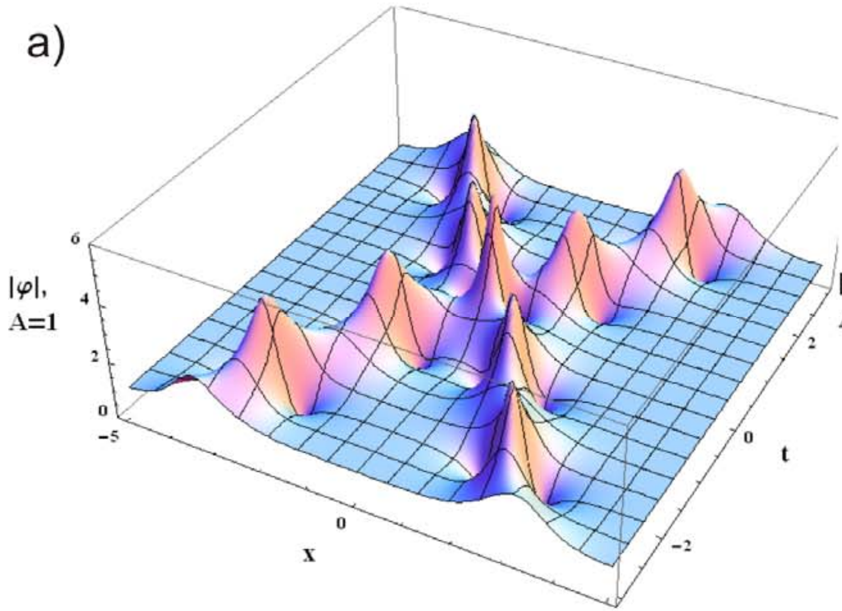
$$\Delta = \frac{(q_1 q_1^*)(q_2 q_2^*)}{(\eta+\eta^*)^2} - \frac{(q_1 q_2^*)(q_1^* q_2)}{4|\eta|^2}$$

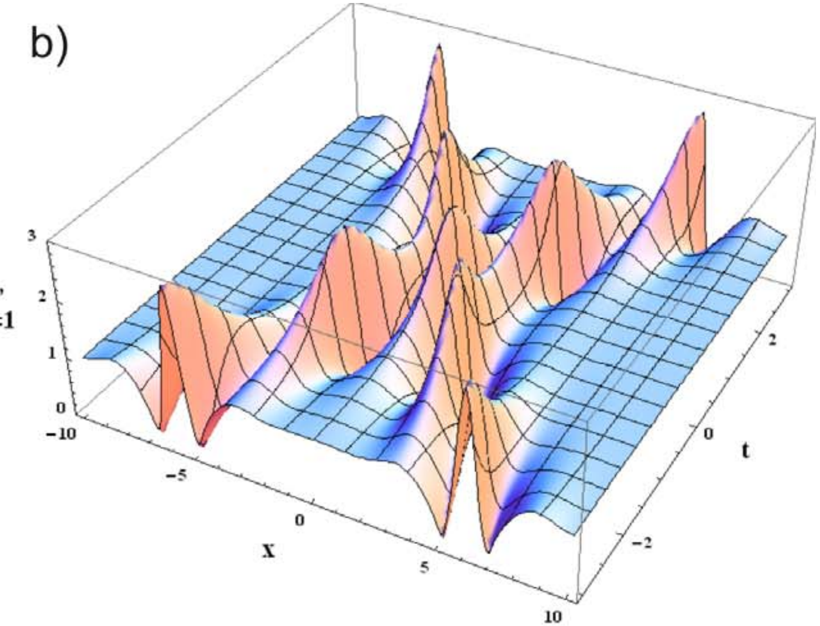
$$q_{1,1}^* = e^{-\varphi_1}\left(1 - \frac{1}{R} e^{-i\alpha} a\right); \qquad q_{1,2}^* = e^{\varphi_1}\left(\frac{1}{R} e^{-i\alpha} + a\right)$$

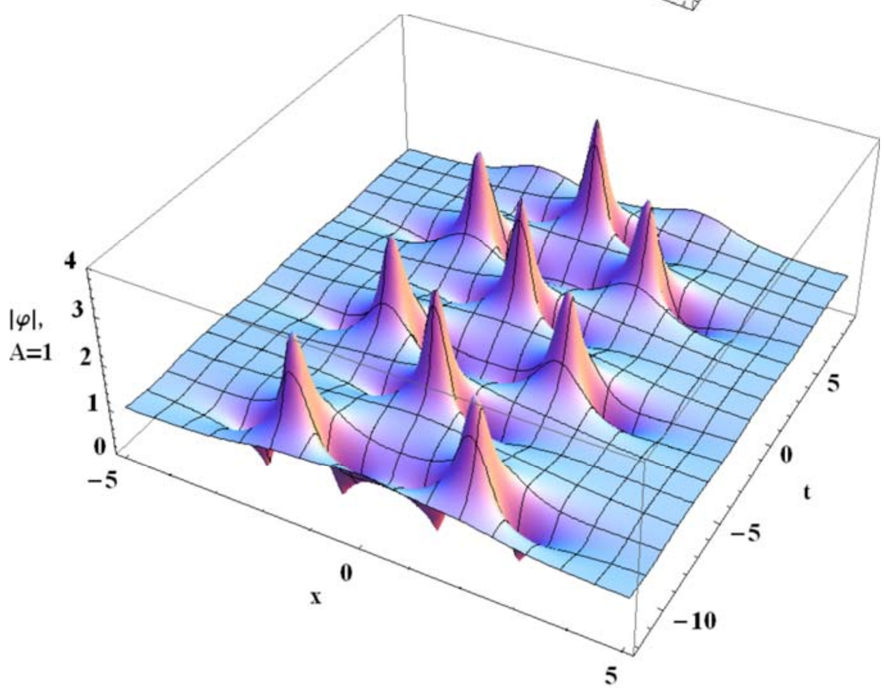
$$q_{2,1}^* = e^{-\varphi_2}\left(1 - \frac{1}{R} e^{i\alpha} b\right); \qquad q_{2,2}^* = e^{\varphi_2}\left(\frac{1}{R} e^{i\alpha} + b\right)$$

$$\varphi_1 = -\frac{1}{2}(æx - \gamma t) + i\frac{i}{2}(kx - \omega t); \quad \varphi_2 = -\frac{1}{2}(æx + \gamma t) - i\frac{i}{2}(kx + \omega t)$$

Two-solitonic solutions

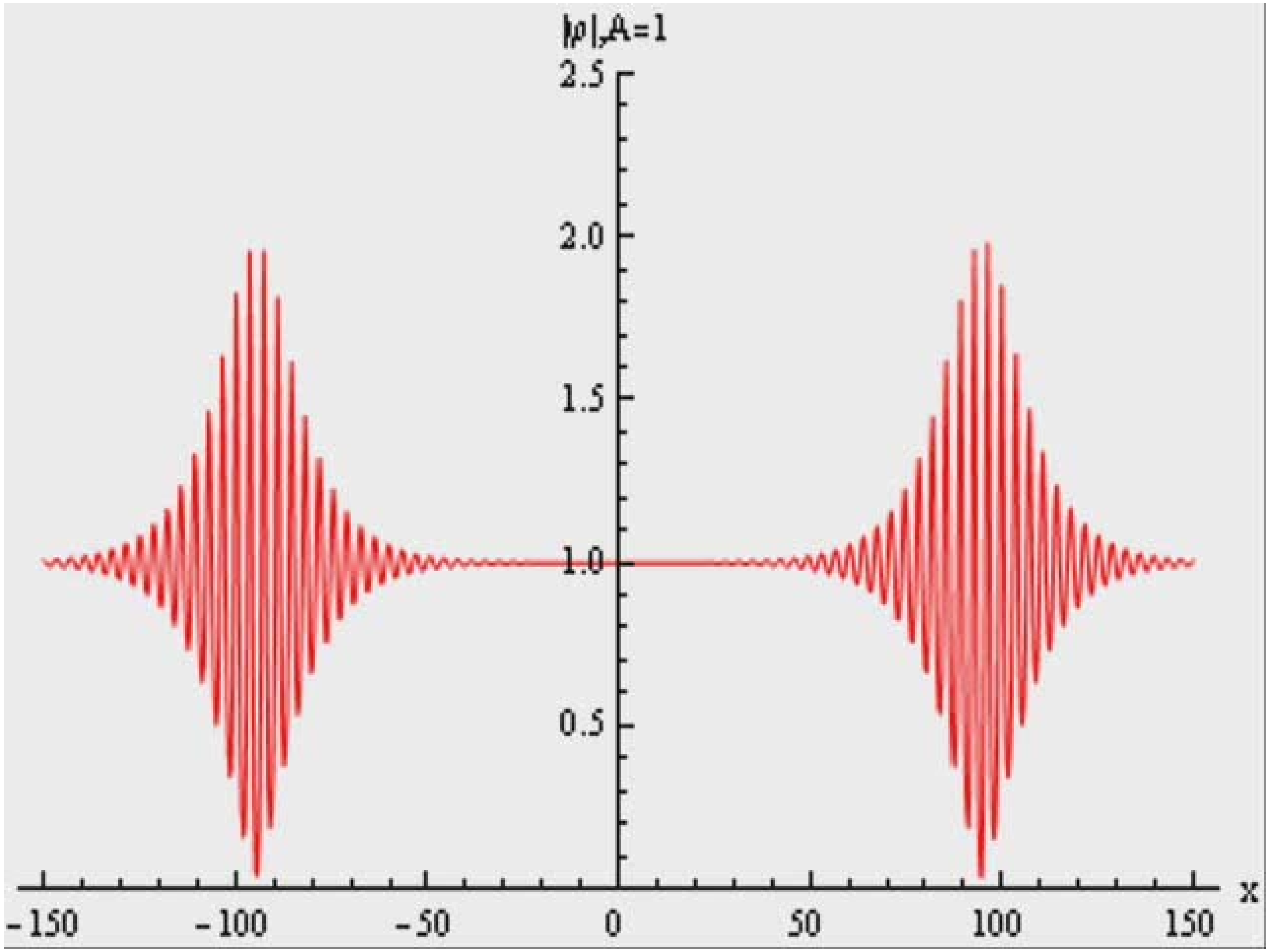




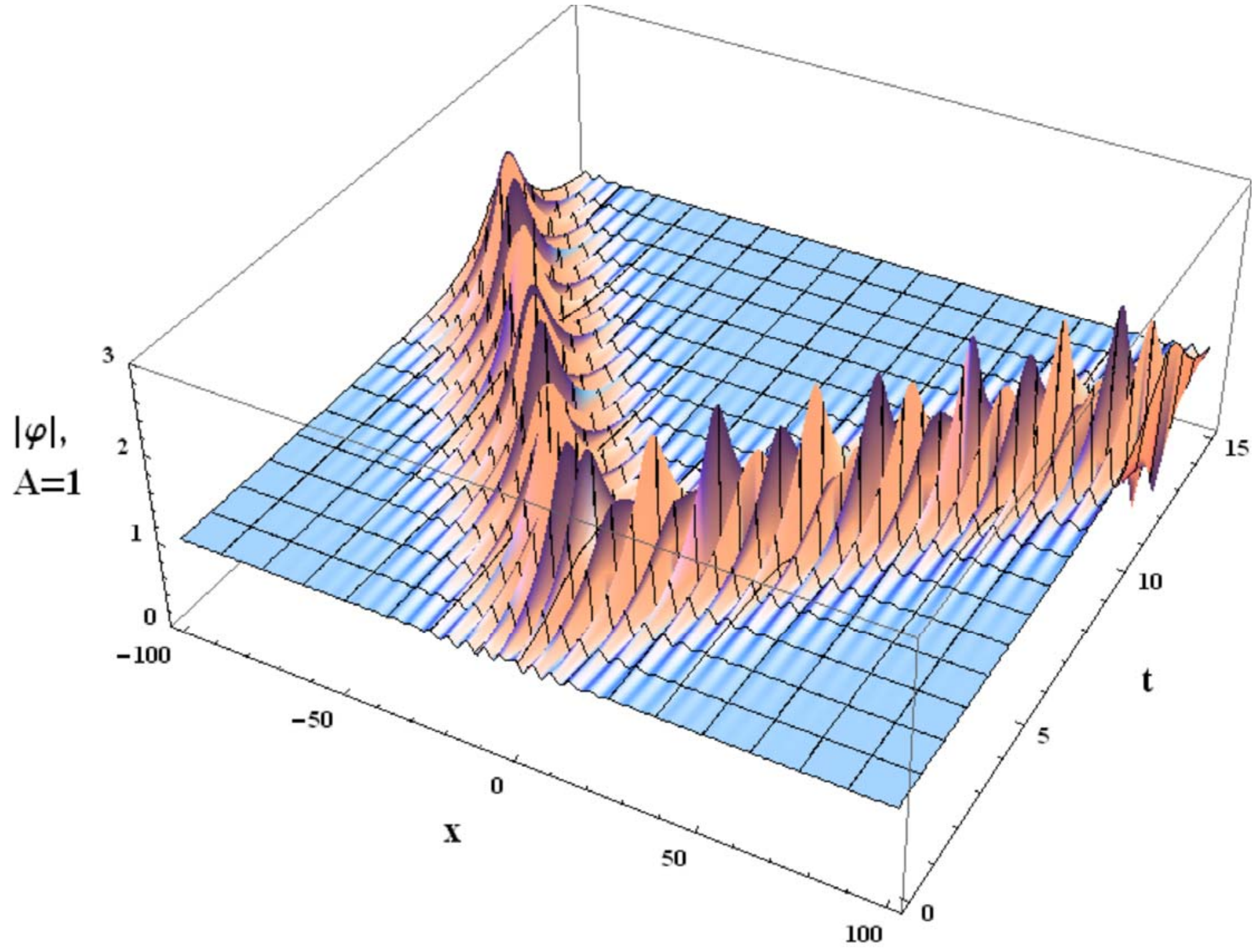


a) R=3, α=π/12
b) R=2, α=π/4
c) R1=1.5, R2=2, α1=0, α2=0

The interference of solitons

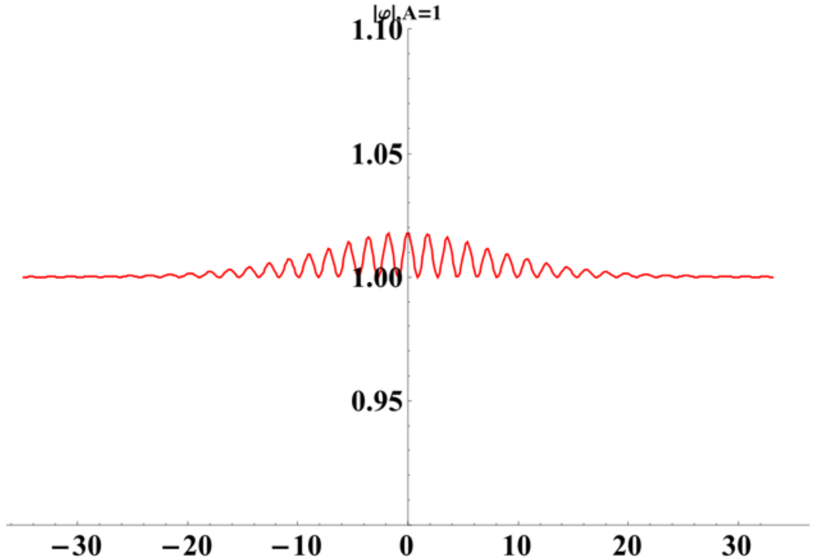


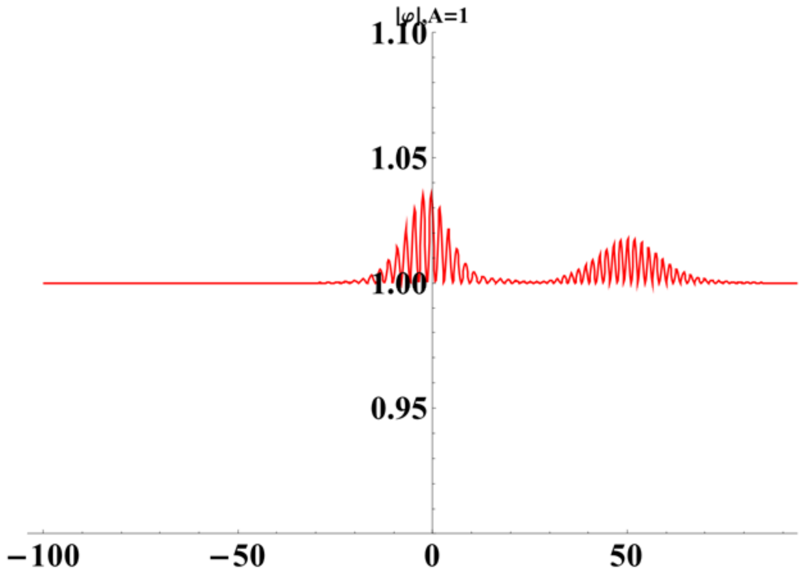
The interference of solitons



R=1.06, α=π/5, a=b=l

A small initial perturbation





$$\varphi = A + D\frac{\sin[2A\sin(\alpha)x-\theta]}{\cosh[2\delta\cos(\alpha)x+S]}\delta$$

Thank you for your attention!