On Canonical Transformation for Water Waves

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Hamiltonian

A one-dimensional potential flow of an ideal incompressible fluid with a free surface in a gravity field fluid is described by the following hamiltonian:

$$H = \frac{1}{2}\int (g\eta^2 + \psi\hat{k}\psi)dx \qquad -\frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx + \ldots$$

here $\eta(x,t)$ - is the shape of a surface, $\phi(x,z,t)$ - is a potential function of the flow and g - is a gravitational constant.

Classical variables Ψ, η

Normal complex variable a_k:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}) \qquad \omega_k = \sqrt{gk}$$

$$\begin{aligned}
\mathcal{H} = & \int \omega_k |a_k|^2 + \int V^k_{k_1 k_2}\{a^*_k a_{k_1} a_{k_2} + a_k a^*_{k_1} a^*_{k_2}\}\delta_{k-k_1-k_2} dk dk_1 dk_2 \\
+ & \frac{1}{3}\int U_{k k_1 k_2}\{a_k a_{k_1} a_{k_2} + a^*_k a^*_{k_1} a^*_{k_2}\}\delta_{k+k_1+k_2} dk dk_1 dk_2 + \\
+ & \frac{1}{2}\int W^{k_2 k_3}_{k k_1} a^*_k a^*_{k_1} a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \\
+ & \frac{1}{3}\int G^{k_4}_{k_1 k_2 k_3}(a^*_{k_1} a^*_{k_2} a^*_{k_3} a_{k_4} + a_{k_1} a_{k_2} a_{k_3} a^*_{k_4} \delta_{k_1+k_2+k_3-k_4} dk_1 dk_2 dk_3 \\
+ & \frac{1}{12}\int R_{k_1 k_2 k_3 k_4}(a^*_{k_1} a^*_{k_2} a^*_{k_3} a^*_{k_4} + a_{k_1} a_{k_2} a_{k_3} a_{k_4})\delta_{k_1+k_2+k_3+k_4} dk_1 dk_2
\end{aligned}$$

Normal variables a_k

a_k satisfies the equation

$$\frac{\partial a_k}{\partial t} + i\frac{\delta H}{\delta a_k^*} = 0,$$

Three wave resonances are absent

$$\begin{aligned} k &= k_1 + k_2, \\ \omega_k &= \omega_{k_1} + \omega_{k_2}, \end{aligned}$$

NO!

Cubic nonresonat terms can be excluded by canonical transformation:

$$a_k \to b_k.$$

Transformation $a_k \to b_k$

$$
\begin{aligned}
a_k &= b_k + \int \Big[-\tilde{V}^{k}_{k_1 k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} + 2\tilde{V}^{k_1}_{k k_2} b_{k_1} b^*_{k_2} \delta_{k_1-k-k_2} - \\
&\quad - \tilde{U}_{k k_1 k_2} b^*_{k_1} b^*_{k_2} \delta_{k+k_1+k_2} \Big] dk_1 dk_2 + \\
&+ \int \big[A^{k}_{k_1 k_2 k_3} b_{k_1} b_{k_2} b_{k_3} + A^{k k_1}_{k_2 k_3} b^*_{k_1} b_{k_2} b_{k_3} + \\
&\quad A^{k k_1 k_2}_{k_3} b^*_{k_1} b^*_{k_2} b_{k_3} + A^{k k_1 k_2 k_3} b^*_{k_1} b^*_{k_2} b^*_{k_3} \big] dk_1 dk_2 dk_3 .
\end{aligned}
$$

Poisson brackets

$$
\{a_{k_1}, a^*_{k_2}\} = \delta(k_1 - k_2), \qquad \{a_{k_1}, a_{k_2}\} = 0
$$

provide conditions for $\tilde{V}^{k}_{k_1 k_2}$, $\tilde{U}_{k k_1 k_2}$ and A.

Transformation $a_k \to b_k$

It is possible to cancel nonresonant both cubic and fourth order terms if

$$\tilde{V}^k_{k_1 k_2} = \frac{V^k_{k_1 k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \qquad \tilde{U}_{k k_1 k_2} = \frac{U_{k k_1 k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}.$$

Classical variables b_k

Coefficients A with upper and lower indices are equal to:

$$
\begin{aligned}
A^{k}_{k_1k_2k_3} &= \left[\frac{1}{3}\tilde{G}^{k}_{k_1k_2k_3} + \tilde{V}^{k}_{k_1k-k_1}\tilde{V}^{k_2+k_3}_{k_2k_3} - \tilde{V}^{k_1}_{kk_1-k}\tilde{U}_{-k_2-k_3k_2k_3}\right]\delta_{k-k_1-k_2} \\
A^{kk_1}_{k_2k_3} &= \Big[\tilde{W}^{k_2k_3}_{kk_1} - 2\tilde{V}^{k}_{k_2k-k_2}\tilde{V}^{k_3}_{k_1k_3-k_1} - \tilde{V}^{k+k_1}_{kk_1}\tilde{V}^{k_2+k_3}_{k_2k_3} + \\
&\quad +2\tilde{V}^{k_3}_{kk_3-k}\tilde{V}^{k_1}_{k_2k_1-k_2} + \tilde{U}_{-k-k_1kk_1}\tilde{U}_{-k_2-k_3k_2k_3}\Big]\delta_{k+k_1-k_2-k_3}, \\
A^{kk_1k_2}_{k_3} &= \Big[-\tilde{G}^{k_3}_{kk_1k_2} + \tilde{V}^{k}_{k_3k-k_3}\tilde{U}_{-k_2-k_1k_2k_1} - \tilde{V}^{k_3}_{kk_3-k}\tilde{V}^{k_1+k_2}_{k_1k_2} + \\
&\quad +2\tilde{V}^{k+k_1}_{kk_1}\tilde{V}^{k_3}_{k_2k_3-k_2} - 2\tilde{U}_{-k-k_1kk_1}\tilde{V}^{k_2}_{k_3k_2-k_3}\Big]\delta_{k+k_1+k_2-k_3}, \\
A^{kk_1k_2k_3} &= \left[-\frac{1}{3}\tilde{R}_{kk_1k_2k_3} - \tilde{V}^{k+k_1}_{kk_1}\tilde{U}_{-k_2-k_3k_2k_3} + \tilde{V}^{k_2+k_3}_{k_2k_3}\tilde{U}_{-k-k_1kk_1}\right]\delta_{k+}
\end{aligned}
$$

Zakharov equation

$$i\dot{b} = \omega_k b_k + \frac{1}{2}\int T_{kk_1}^{k_2k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

Zakharov equation

$$i\dot{b} = \omega_k b_k + \frac{1}{2}\int T_{kk_1}^{k_2 k_3} b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3$$

$$\begin{aligned}
T_{k_2 k_3}^{k k_1} &= W_{k_1 k}^{k_2 k_3} - \\
&- V_{k_2\, k-k_2}^{k} V_{k_1\, k_3-k_1}^{k_3} \left[\frac{1}{\omega_{k_2} + \omega_{k-k_2} - \omega_k} + \frac{1}{\omega_{k_1} + \omega_{k_3-k_1} - \omega_{k_3}} \right] - \\
&- V_{k_2\, k_1-k_2}^{k_1} V_{k\, k_3-k}^{k_3} \left[\frac{1}{\omega_{k_2} + \omega_{k_1-k_2} - \omega_{k_1}} + \frac{1}{\omega_{k} + \omega_{k_3-k} - \omega_{k_3}} \right] - \\
&- V_{k_3\, k-k_3}^{k} V_{k_1\, k_2-k_1}^{k_2} \left[\frac{1}{\omega_{k_3} + \omega_{k-k_3} - \omega_k} + \frac{1}{\omega_{k_1} + \omega_{k_2-k_1} - \omega_{k_2}} \right] - \\
&- V_{k_3\, k_1-k_3}^{k_1} V_{k\, k_2-k}^{k_2} \left[\frac{1}{\omega_{k_3} + \omega_{k_1-k_3} - \omega_{k_1}} + \frac{1}{\omega_{k} + \omega_{k_2-k} - \omega_{k_2}} \right] - \\
&- V_{k k_1}^{k+k_1} V_{k_2 k_3}^{k_2+k_3} \left[\frac{1}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} + \frac{1}{\omega_{k_2+k_3} - \omega_{k_2} - \omega_{k_3}} \right] - \\
&- U_{-k-k_1\, k k_1} U_{-k_2-k_3\, k_2 k_3} \left[\frac{1}{\omega_{k+k_1} + \omega_k + \omega_{k_1}} + \frac{1}{\omega_{k_2+k_3} + \omega_{k_2} + \omega_{k_3}} \right]
\end{aligned}$$

$T_{kk_1}^{k_2k_3}$ ***vanishes***

On the resonant manifolld

$$k + k_1 = k_2 + k_3, \qquad \omega_k + \omega_{k_1} = \omega_{k_2} + \omega_{k_3},$$

$$\begin{aligned} k &= a(1+\zeta)^2, \\ k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \\ k_3 &= a(1+\zeta+\zeta^2)^2; \end{aligned}$$

here $0 < \zeta < 1$ and $a > 0$.

$$T_{kk_1}^{k_2k_3} = 0!$$

If so, it is possible to simplify four-wave interactions !

Canonical transformation for b_k**,** $k > 0$

Using this diagonal part ($\hat{T}_{kk_1}$), one can construct the following function:

$$\tilde{T}^{kk_1}_{k_2k_3} = \left[\frac{1}{2}(T_{kk_2} + T_{kk_3} + T_{k_1k_2} + T_{k_1k_3}) - \frac{1}{4}(T_{kk} + T_{k_1k_1} + T_{k_2k_2} + T_{k_3k_3})\right]\theta(kk_1k_2k_3)$$

$\tilde{T}^{kk_1}_{kk_1}$ coincides with original four-wave coefficient on the resonant manifold. Choose $\tilde{W}^{kk_1}_{k_2k_3}$ as follow

$$\tilde{W}^{kk_1}_{k_2k_3} = \frac{\tilde{T}^{kk_1}_{k_2k_3} - T^{kk_1}_{k_2k_3}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}}$$

$$\mathcal{H} = \int \omega_k b_k b_k^* dk + \frac{1}{2}\int \tilde{T}^{k_2k_3}_{kk_1} b_k^* b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk dk_1 dk_2 dk_3 + \ldots$$

Compact Hamiltonian

$$\mathcal{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{2} \int |b'|^2 \left[\frac{i}{2}(bb'^* - b^* b') - \hat{K}|b|^2 \right] dx.$$

Corresponding dynamical equation is

$$\begin{aligned} i\frac{\partial b}{\partial t} = \hat{\omega}_k b \quad &+ \quad \frac{i}{4}\left[b^* \frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime} \frac{\partial}{\partial x} b^2) \right] \\ &- \quad \frac{1}{2}\left[b \cdot \hat{K}(|b'|^2) - \frac{\partial}{\partial x}(b' \hat{K}(|b|^2)) \right]. \end{aligned}$$

Transformation from b_k to η_k and ψ_k

$$
\begin{aligned}
\eta(x) &= \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{\frac{1}{4}}b(x) + \hat{k}^{\frac{1}{4}}b(x)^*) + \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{\frac{1}{4}}b(x) - \hat{k}^{\frac{1}{4}}b^*(x)]^2 + O(b^3 \\
\psi(x) &= -i\frac{g^{\frac{1}{4}}}{\sqrt{2}}(\hat{k}^{-\frac{1}{4}}b(x) - \hat{k}^{-\frac{1}{4}}b(x)^*) + \\
&+ \frac{i}{2}[\hat{k}^{\frac{1}{4}}b^*(x)\hat{k}^{\frac{3}{4}}b^*(x) - \hat{k}^{\frac{1}{4}}b(x)\hat{k}^{\frac{3}{4}}b(x)] + \\
&+ \frac{1}{2}\hat{H}[\hat{k}^{\frac{1}{4}}b(x)\hat{k}^{\frac{3}{4}}b^*(x) + \hat{k}^{\frac{1}{4}}b^*(x)\hat{k}^{\frac{3}{4}}b(x)] + O(b^3)
\end{aligned}
$$

$O(b^3)$ are defined by all A with lower and upper indexes:

A

$$A^{k_1}_{k_2k_3k_4} = +\frac{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}}{48\pi g}k_1(k_1k_2k_3k_4)^{\frac{1}{4}}$$

$$A^{-k_1k_2k_3k_4} = \frac{\omega_{k_1}-\omega_{k_2}-\omega_{k_3}-\omega_{k_4}}{48\pi g}k_1(k_1k_2k_3k_4)^{\frac{1}{4}}$$

$$A^{k_1k_2k_3}_{k_4} = \frac{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}}{16\pi g}k_1(k_1k_2k_3k_4)^{\frac{1}{4}}.$$

Conclusions

$$H = \frac{1}{2}\int (g\eta^2 + \psi\hat{k}\psi)dx \qquad -\frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + $$
$$+\frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx + \ldots$$

$$\mathcal{H} = \int b^*\hat{\omega}_k b dx + \frac{1}{2}\int |b'|^2 \left[\frac{i}{2}(bb'^* - b^*b') - \hat{K}|b|^2\right] dx.$$

$$\begin{aligned} \eta(x) &= \Pi(b(x), b^*(x)) \\ \psi(x) &= \Psi(b(x), b^*(x)) \end{aligned}$$

$$\left[\eta(x) = \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{\frac{1}{4}}b(x) + \hat{k}^{\frac{1}{4}}b(x)^*) + \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{\frac{1}{4}}b(x) - \hat{k}^{\frac{1}{4}}b^*(x)]^2\right]$$