Dynamics of two-dimensional dark quasisolitons in a smoothly inhomogeneous Bose-Einstein condensate

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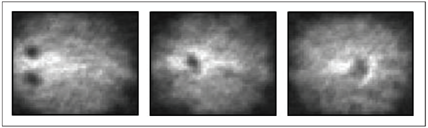
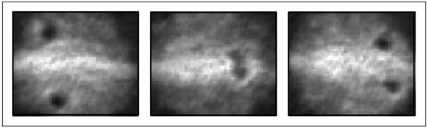
ИПФ РАН

[1] *Neely T. W., Samson E. C., Bradley A. S., Davis M. J., Anderson B. P.* Observation of Vortex Dipoles in an Oblate Bose-Einstein Condensate. // Phys. Rev. Lett. 2010. Vol. 104, no. 16. Pp. 160401 (4).

BEC of ^{87}Rb atoms, $N_{atoms} = (2 \pm 0.5) \times 10^6$, $T = 52\,\text{nK}$,

$(\omega_r, \omega_z) = 2\pi \times (8, 90)\text{Hz}$, $R_{TF,z} = 5\,\mu\text{m}$, $R_{TF,r} = 52\,\mu\text{m}$.

(1) 

(2) 

[2] *Freilich D. V., Bianchi D. M., Kaufman A. M., Langin T. K., Hall D. S.* Real-Time Dynamics of Single Vortex Lines and Vortex Dipoles in a Bose-Einstein Condensate. // Science. 2010. Vol. 329, no. 5996. Pp. 1182-1185.

[3] *Middelkamp S., Torres P. J., Kevrekidis P. G., Frantzeskakis D. J., Carretero-Gonzàlez R., Schmelcher P., Freilich D. V., Hall D. S.* Guiding-center dynamics of vortex dipoles in Bose-Einstein condensates. // Phys. Rev. A. 2011. Vol. 84, no. 1. Pp. 011605(R) (4).

1. Basic equation of the BEC dynamics.
2. 2D dark solitons in a homogeneous BEC.
3. Asymptotic description of the dynamics of 2D dark quasisolitons in a smoothly inhomogeneous BEC.
4. Geometric optics of 2D dark quasisolitons.
5. Numerical simulation of the dynamics of 2D dark quasisolitons in a smoothly inhomogeneous BEC.
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Gross-Pitaevskii (GP) equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi + g\left|\Psi\right|^2\Psi + V_{ext}\left(\mathbf{r}\right)\Psi.$$

Here, m is the atomic mass;

a_s is the s-wave scattering length;

$g = 4\pi\hbar^2 a_s/m$ is the interaction coupling constant;

$\Psi(\mathbf{r},t) = |\Psi(\mathbf{r},t)|\exp\left(i\vartheta(\mathbf{r},t)\right)$ is the classical wave function;

$n(\mathbf{r},t) = |\Psi(\mathbf{r},t)|^2$ is the density of condensate atoms;

$\mathbf{v}(\mathbf{r},t) = \left(\hbar/m\right)\nabla\vartheta(\mathbf{r},t)$ is the velocity field in the BEC;

$V_{ext}\left(\mathbf{r}\right)$ is the potential of the external forces acting on the BEC.

- $a_s > 0$ for repulsive interactions between atoms.

Dimensionless variables: $\mathbf{r}' = \mathbf{r}/r_0, \quad t' = t/t_0, \quad \mathbf{v}' = \mathbf{v}/c_s.$

$$\Psi' = \Psi \exp(-it')/\sqrt{n_0}, \quad V'_{ext} = V_{ext}/gn_0.$$

$r_0 = \hbar/\sqrt{mgn_0}$ is the healing length; $t_0 = r_0/c_s = \hbar/gn_0$;
$c_s = \sqrt{gn_0/m}$ is the sound velocity.

Nonlinear Schrödinger (NLS) equation:

$$\boxed{i\frac{\partial\Psi}{\partial t} + \frac{1}{2}\Delta\Psi + \left(1 - |\Psi|^2\right)\Psi = V_{ext}(\mathbf{r})\,\Psi.}$$

Madelung transform: $\Psi(\mathbf{r},t) = \psi(\mathbf{r},t)\exp\left(i\theta(\mathbf{r},t)\right).$

System of hydrodynamic equations for a compressible inviscid liquid:

$$\partial_t\psi^2 + \mathrm{div}\left(\psi^2\nabla\theta\right) = 0,$$
$$\partial_t\theta + (\nabla\theta)^2/2 = 1 - \psi^2 + \Delta\psi/2\psi.$$

- We restricted our consideration to the 2D problem: $\mathbf{r} = (x, y)$.

The GP equation with $V_{ext}(\mathbf{r})=0$ has a single-parameter family of solutions in the form of 2D dark solitons moving at a constant velocity $\bar{v}$:

$$\Psi_s=\Psi_s(\xi,y,\bar{v}), \quad \Psi_s\left(\sqrt{\xi^2+y^2}\to\infty\right)\to 1, \quad \xi=x-\bar{v}t, \quad \bar{v}=const.$$

$$-i\bar{v}\frac{\partial\Psi_s}{\partial\xi}+\frac{1}{2}\frac{\partial^2\Psi_s}{\partial\xi^2}+\frac{1}{2}\frac{\partial^2\Psi_s}{\partial y^2}+\left(1-|\Psi_s|^2\right)\Psi_s=0.$$

Here, the velocity $\bar{v}$ plays the role of the problem parameter. Depending on $\bar{v}$, 2D dark solitons can be "vortex" and "vortex-free":

- the vortex pairs are implemented for $0<|\bar{v}|<\bar{v}_*$;
- the vortex-free solitons are implemented for $\bar{v}_*<|\bar{v}|<1$;
- when $|\bar{v}|=\bar{v}_*\approx 0.61$, there is a continuous transition from the vortex state with two zeroes of the density into a vortex-free state, in which the density is different from zero everywhere.

The properties of the 2D dark solitons were studied in detail in the references:

[1] *Jones C. A., Roberts H.* Motions in a Bose condensate. IV. Axisymmetric solitary waves. // J. Phys. A: Math. Gen. 1982. Vol. 15, no. 8. Pp. 2599–2619.

[2] *Jones C. A., Putterman S., Roberts H.* Motions in a Bose condensate. V. Stability of solitary wave solutions of non-linear Schrodinger equations in two and three dimensions. // J. Phys. A: Math. Gen. 1986. Vol. 19, no. 15. Pp. 2991–3011.

[3] *Kuznetsov E. A., Rasmussen J. J.* Instability of two-dimensional solitons and vortices in defocusing media. // Phys. Rev. E. 1995. Vol. 51, no. 5. Pp. 4479-4484.

[4] *Berloff N. G.* Padé approximations of solitary wave solutions of the Gross-Pitaevskii equation. // J. Phys. A: Math. Gen. 2004. Vol. 37, no. 5. Pp. 1617–1632.

[5] *Tsuchiya S., Dalfovo F., Pitaevskii L. P.* Solitons in two-dimensional Bose-Einstein condensates. // Phys. Rev. A. 2008. Vol. 77, no. 4. Pp. 045601 (4).

Variation problem: $\delta\left(\mathcal{H} - \bar{v}\mathcal{P}_x\right) = 0.$

Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}\xi \left(|\nabla\Psi|^2 + \left(1 - |\Psi|^2\right)^2\right).$$

Momentum:

$$\mathbf{P} = \frac{i}{2}\int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}\xi \left(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi\right).$$

Integral relations for 2D dark solitons:

(1) $$\bar{\mathcal{E}} \equiv \mathcal{H}_s = \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}\xi \left|\frac{\partial \Psi_s}{\partial \xi}\right|^2,$$

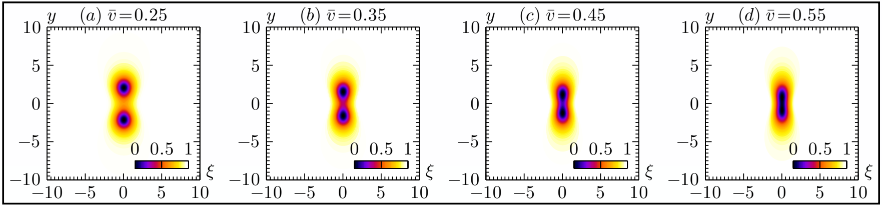
(2) $$\bar{\mathcal{E}} - \bar{v}\bar{\mathcal{P}} = \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}\xi \left|\frac{\partial \Psi_s}{\partial y}\right|^2,$$

(3) $$\bar{v}\bar{\mathcal{P}} = \int_{-\infty}^{+\infty} \mathrm{d}y \int_{-\infty}^{+\infty} \mathrm{d}\xi \left(1 - |\Psi_s|^2\right)^2.$$

The velocity $\bar{v}$ of a 2D dark soliton is defined as

$$\bar{v} = \frac{\mathrm{d}\bar{\mathcal{E}}}{\mathrm{d}\bar{\mathcal{P}}} < \frac{\bar{\mathcal{E}}}{\bar{\mathcal{P}}}.$$

Vortex 2D dark solitons.



The limit of small velocities

$|\bar{v}| \ll 1 \Rightarrow \bar{\mathcal{E}} \gg 1.$

In this limit 2D dark soliton solutions are vortex pairs, whose wave function is well described by the expression

$$\Psi_{vp}(\xi, y, \bar{v}) = \psi_{vp}(\xi, y, \bar{v}) \exp\left(i\theta_{vp}(\xi, y, \bar{v})\right),$$

where $\theta_{vp}(\xi, y, \bar{v}) = \varphi_1(\xi, y) - \varphi_2(\xi, y)$ is the phase of the vortex pair,

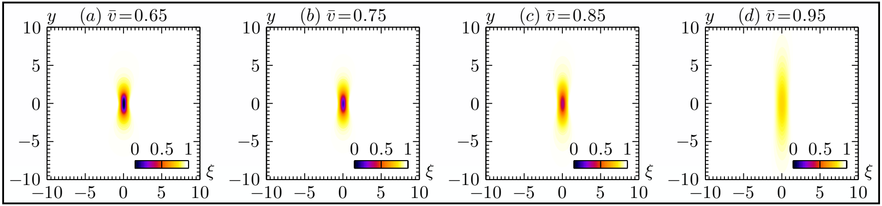
$\varphi_{1(2)}(\xi, y)$ are the polar angles around the points $\big(\xi = 0, y = \pm l/2\big)$,

$l = l(\bar{v})$ is the distance between topological defects (zeroes of the density).

The energy $\bar{\mathcal{E}}$, momentum $\bar{\mathcal{P}}$, and velocity $\bar{v}$ are related with l as for a pairs of point vortices:

$$\bar{\mathcal{E}} \approx 2\pi \ln(l), \quad \bar{\mathcal{P}} \approx 2\pi l, \quad \bar{v} \approx 1/l. \quad \Rightarrow \quad \bar{\mathcal{P}}\big(\bar{\mathcal{E}} \gg 1\big) = 2\pi \exp\big(\bar{\mathcal{E}}/2\pi\big).$$

Vortex-free 2D dark solitons.



The limit of transonic velocities

$(1-|\bar{v}|)\ll 1 \Rightarrow \bar{\mathcal{E}}\ll 1.$ (The value of $|\bar{v}|$ tends to the sound velocity: $|\bar{v}|\to 1$.)

In this limit 2D dark soliton solutions coincide with known solutions of the Kadomtsev-Petviashvili (KP) equation.

Weakly nonlinear limit:

$$T=\epsilon^{3/2}t,$$

$$X=\epsilon^{1/2}(x-t),\quad Y=\epsilon y,$$

$$\theta=\epsilon^{(j-1/2)}\theta_j,\quad n=1+\epsilon^j n_j.$$

$$\Downarrow$$

$$\bar{\mathcal{P}}(\bar{\mathcal{E}}\ll 1)\approx\bar{\mathcal{E}}-\frac{3}{128\pi^2}\bar{\mathcal{E}}^3.$$

$\Rightarrow$

Kadomtsev-Petviashvili (KP) equation:

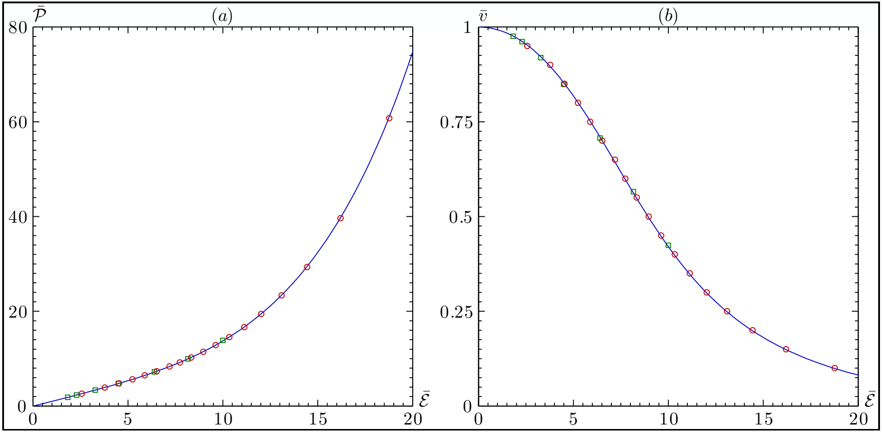
$$\frac{\partial}{\partial X}\left(2\frac{\partial n_1}{\partial T}+3n_1\frac{\partial n_1}{\partial X}-\frac{1}{4}\frac{\partial^3 n_1}{\partial X^3}\right)=-\frac{\partial^2 n_1}{\partial Y^2}.$$

$$n_{s,1}\left(\xi,y,\bar{v}=\left(1-\tilde{v}_{kp}\right)\right)=\epsilon n_1=$$

$$=\frac{16\tilde{v}_{kp}\left[3-8\tilde{v}_{kp}\xi^2+16\tilde{v}_{kp}^2y^2\right]}{\left[3+8\tilde{v}_{kp}\xi^2+16\tilde{v}_{kp}^2y^2\right]^2}.$$

Analytical approximation for the dependence $\bar{\mathcal{P}}(\bar{\mathcal{E}})$:

$$\bar{\mathcal{P}}(\bar{\mathcal{E}}) = \alpha(\bar{\mathcal{E}}) \sinh\left[\frac{\bar{\mathcal{E}}}{\alpha(\bar{\mathcal{E}})}\right], \qquad \alpha(\bar{\mathcal{E}}) = 2\pi + \frac{2\pi}{3} \exp\left[-\frac{\bar{\mathcal{E}}^2}{\sigma_{\bar{\mathcal{E}}}^2}\right], \qquad \sigma_{\bar{\mathcal{E}}} = 9.8.$$



III. Asymptotic description of the dynamics of 2D dark quasisolitons in a smoothly inhomogeneous BEC

$$\Psi(\mathbf{r},t) = G(\mathbf{r})\,F(\mathbf{r},t)\,.$$

$G(\mathbf{r})$ is the undisturbed part, which is formed under the action of the potential $V_{ext}(\mathbf{r})$.

$$\frac{1}{2}\Delta G + \left(1 - |G|^2\right)G - V_{ext}(\mathbf{r})\,G = 0.$$

$$G(\mathbf{r}) = g(\mathbf{r})\exp(i\theta_g), \qquad \theta_g = const.$$

$\Lambda_G \sim |G| / |\nabla G|$ is a characteristic scale of the wave function $G(\mathbf{r})$ of the undisturbed condensate.

$\Lambda_G \gg r_c.$

Thomas-Fermi approximation:

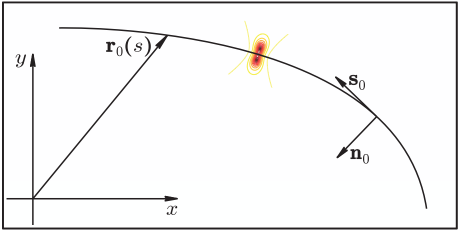
$$n_{g_{TF}}(\mathbf{r}) = g^2_{TF}(\mathbf{r}) = \left(1 - V_{ext}(\mathbf{r})\right) > 0.$$

$F(\mathbf{r},t)$ is the part of the wave function $\Psi(\mathbf{r},t)$ describing the behavior of the finite-amplitude disturbances in an initially inhomogeneous BEC.

$$i\frac{\partial F}{\partial t} + \frac{1}{2}\Delta F + |G|^2\left(1 - |F|^2\right)F = -\left(\frac{\nabla G}{G}, \nabla F\right).$$

Λ_{qs} is a characteristic size of the localization region of the considered disturbance initially specified in the form of a 2D quasisoliton structure.

Small parameter: $\mu = \Lambda_{qs}/\Lambda_G \ll 1.$



A schematic representation of propagation of a 2D dark quasisoliton moving along the trajectory $\mathbf{r}_0(s)$ in a smoothly inhomogeneous BEC.

Here, $\mathbf{s}_0$ and $\mathbf{n}_0$ are the unit vectors of the tangent and the normal to the propagation path $\mathbf{r}_0(s)$, respectively;

s is the arc length;

η is the distance along the normal dropped on the curve $\mathbf{r}_0(s)$.

$$\mathbf{r} = \mathbf{r}_0(s) + \eta\,\mathbf{n}_0(s), \qquad \eta = \eta\,(x,y), \qquad s = s\,(x,y).$$

This equation relates the Cartesian coordinates (x, y) with the orthogonal curvilinear coordinates (s, η), the transition to which is characterized by the following Lamé coefficients: $h_s = (1 - \kappa\eta), \quad h_\eta = 1,$

where $\kappa = \kappa(s)$ is the curvature of the line $\mathbf{r}_0(s)$, $\quad \kappa \sim \mu$.

We rewrote the equation for the function $F(\mathbf{r}, t)$ in terms of the variables s and η:

$$i\frac{\partial F}{\partial t} + \frac{1}{2h_s h_\eta}\left[\frac{\partial}{\partial s}\left(\frac{h_\eta}{h_s}\frac{\partial F}{\partial s}\right) + \frac{\partial}{\partial \eta}\left(\frac{h_s}{h_\eta}\frac{\partial F}{\partial \eta}\right)\right] + |G|^2\left(1 - |F|^2\right)F =$$

$$= -\frac{1}{G}\left[\frac{1}{h_s^2}\frac{\partial G}{\partial s}\frac{\partial F}{\partial s} + \frac{1}{h_\eta^2}\frac{\partial G}{\partial \eta}\frac{\partial F}{\partial \eta}\right].$$

We used the assumption that the characteristic variation scale Λ_G of the function $G(\mathbf{r})$ significantly exceeds the sizes Λ_{qs} of the localization region of the quasisoliton. Expanding the function $|G(\mathbf{r})|^2$ near the curve $\mathbf{r}_0(s)$ into a Taylor series of η, namely,

$$|G(\mathbf{r})|^2 = |G(\mathbf{r}_0 + \eta \mathbf{n}_0)|^2 \approx |G(\mathbf{r}_0)|^2 + \left[\frac{\partial |G|^2}{\partial \eta}\right]\Bigg|_{\eta=0} \eta + \dots .$$

For the localized quasisoliton formation moving along the trajectory $\mathbf{r}_0(s)$, it is convenient to pass from the arc length s to the coordinate ξ “accompanying” the quasisoliton:

$$\xi = s - s_{qs}(t), \quad s_{qs}(t) = \int_0^t v(t)\, \mathrm{d}t,$$

where $s_{qs}(t)$ is the position of the center of a quasisoliton on the curve $\mathbf{r}_0(s)$, $v(t)$ is the velocity of a 2D dark quasisoliton.

After such a transition, the solution of the equation for the function $F(\mathbf{r}, t)$ near the curve $\mathbf{r}_0(s)$ can be represented as an asymptotic series of μ:

$$F(\xi, \eta, t) = \mu^0 F_0(\xi, \eta, v(\mu t)) + \mu^1 F_1(\xi, \eta, \mu t) + \mu^2 F_2(\xi, \eta, \mu t) + \dots .$$

- μ^0: $$-iv \frac{\partial F_0}{\partial \xi} + \frac{1}{2}\frac{\partial^2 F_0}{\partial \xi^2} + \frac{1}{2}\frac{\partial^2 F_0}{\partial \eta^2} + g^2(s_s, 0)\left(1 - |F_0|^2\right) F_0 = 0.$$

$$\bar{\xi} = g(s_s, 0)\, \xi, \quad \bar{\eta} = g(s_s, 0)\, \eta, \quad \bar{v} = v / g(s_s, 0). \qquad \Rightarrow \qquad F_0(\xi, \eta, v) \equiv \Psi_s\left(\bar{\xi}, \bar{\eta}, \bar{v}\right).$$

$(s_{qs},0)$ is the position of the center of the 2D dark quasisoliton, which is determined in the coordinate system (s,η).

$g^2(s_{qs},0)$ is the density of the undisturbed inhomogeneous BEC at the point $(s_{qs},0)$.

- μ^1: $$-i\bar{v}\frac{\partial F_1}{\partial \bar{\xi}}+\frac{1}{2}\frac{\partial^2 F_1}{\partial \bar{\xi}^2}+\frac{1}{2}\frac{\partial^2 F_1}{\partial \bar{\eta}^2}+g^2(s_s,0)\left(\left(1-2\left|F_0\right|^2\right)F_1-F_0^2F_1^*\right)=\mathcal{R}.$$

$$\mathcal{R}=-\left(\xi\left[\frac{\partial g^2}{\partial s}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}+\eta\left[\frac{\partial g^2}{\partial \eta}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}\right)\left(1-|F_0|^2\right)F_0-\frac{i}{2}\frac{\mathrm{d}v^2}{\mathrm{d}s_{qs}}\frac{\partial F_0}{\partial v}-\kappa\eta\frac{\partial^2 F_0}{\partial \xi^2}-$$
$$-\left(iv\xi+1\right)\left[\frac{\partial \ln g}{\partial s}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}\frac{\partial F_0}{\partial \xi}-\left(iv\eta\left[\frac{\partial \ln g}{\partial s}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}+\left[\frac{\partial \ln g}{\partial \eta}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}-\frac{\kappa}{2}\right)\frac{\partial F_0}{\partial \eta}.$$

The conditions for the existence of localized (both in ξ and η) solutions in the equation for the function F_1 are the fulfillment of the following equalities:

$$\mathrm{Re}\left[\int_{-\infty}^{+\infty}\mathrm{d}\eta\int_{-\infty}^{+\infty}\mathrm{d}\xi\,\mathcal{R}\,\frac{\partial F_0^*}{\partial \zeta_{1(2)}}\right]=0,\qquad \zeta_1\equiv\xi,\qquad \zeta_2\equiv\eta.$$

These relations for the complex equation for the function F_1 essentially counterparts of the Fredholm theorem on alternative.

Using the symmetry of considered localized structures, namely, $\Psi_s\left(\bar{\xi},\bar{\eta},\bar{v}\right)=\Psi_s\left(\bar{\xi},-\bar{\eta},\bar{v}\right)$, and the integral relations for 2D dark solitons, as a result, we arrived at the following equations:

$$\frac{\mathrm{d}\ln\bar{\mathcal{E}}}{\mathrm{d}s_{qs}} = -2\left[\frac{\partial\ln g}{\partial s}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}, \qquad \kappa = \left[\frac{\partial\ln g}{\partial\eta}\right]\Bigg|_{\substack{s=s_{qs}\\ \eta=0}}\left(1-\frac{2\bar{\mathcal{E}}}{\bar{v}\bar{\mathcal{P}}}\right).$$

The quantities $\bar{\mathcal{P}}=\bar{\mathcal{P}}(\bar{v}(t))$ and $\bar{\mathcal{E}}=\bar{\mathcal{E}}(\bar{v}(t))$ at each moment of time t are the corresponding characteristics of a 2D dark soliton propagating with the velocity $\bar{v}(t)$ for a given value of $\bar{v}$ in a homogeneous condensate of unit density.

Hereafter we call $\bar{\mathcal{P}}(\bar{v}(t))$ and $\bar{\mathcal{E}}(\bar{v}(t))$ the normalized momentum and the normalized energy of a 2D dark quasisoliton, respectively.

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1. Dependence of the normalized energy of a 2D dark quasisoliton on its position along the propagation path

$$\bar{\mathcal{E}}(s_s) = \frac{\bar{\mathcal{E}}(0)\, g^2(0,0)}{g^2(s_s,0)}.$$

Here, $\bar{\mathcal{E}}(0)$ are the initial values of the normalized energy, $g^2(0,0)$ is the density of the undisturbed inhomogeneous condensate at the point $(s_{qs}(t=0)=0,0)$, from which the 2D dark quasisoliton starts.

The normalized momentum $\bar{\mathcal{P}}$ and the normalized velocity $\bar{v}$ are single-valued functions of the normalized energy $\bar{\mathcal{E}}$; hence, both $\bar{\mathcal{P}}$ and $\bar{v}$ are functions of only $g(s_{qs},0)$.

$$\bar{\mathcal{P}}=\bar{\mathcal{P}}\big(\bar{\mathcal{E}}\big), \quad \bar{v}=\bar{v}\big(\bar{\mathcal{E}}\big). \quad \Rightarrow \quad \bar{\mathcal{P}}=\bar{\mathcal{P}}\big(g(s_s,0)\big), \quad \bar{v}=\bar{v}\big(g(s_s,0)\big).$$

It follows that an increase (decrease) in the density $n_g(s_{qs},0) = g^2(s_{qs},0)$ of the undisturbed condensate along the propagation path leads to a consistent decrease (increase) in $\bar{\mathcal{E}}(s_{qs})$ and $\bar{\mathcal{P}}(s_{qs})$ and an increase (decrease) in $\bar{v}(s_{qs})$.

As a result, the vortex pair as it penetrates into the more dense condensate may become a vortex-free soliton, and vice versa the vortex-free soliton as it reaches the less dense Bose gas may convert into a vortex pair.

2. Equation of the 2D dark quasisoliton trajectory

The expression for the curvature κ uniquely specify on the plane (x, y) the trajectory $\mathbf{r}_0(s)$, along which the quasisoliton moves:

$$\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s} = \mathbf{s}_0(s), \qquad \frac{\mathrm{d}\mathbf{s}_0}{\mathrm{d}s} = \kappa(s)\mathbf{n}_0(s), \qquad \kappa(s) = \left[\left(1 - \frac{2\bar{\mathcal{E}}(g)}{\bar{v}(g)\bar{\mathcal{P}}(g)}\right)\frac{\partial \ln g}{\partial \eta}\right]\Bigg|_{\eta=0}.$$

Here $\mathbf{s}_0(s)$ and $\mathbf{n}_0(s)$ are the unit vectors of the tangent and the normal to the curve $\mathbf{r}_0(s)$, respectively.

By analogy with geometrical optics, we introduced, instead of the arc length s, a new variable τ:

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{\nu\big(g(s,0)\big)}, \qquad \nu\big(g(s,\eta)\big) = \frac{g(s,\eta)\,\bar{\mathcal{P}}\big(g(s,\eta)\big)}{\bar{\mathcal{E}}(0)\,g^2(0,0)},$$

where the function $\nu\big(g(s,\eta)\big)$ has the meaning of an "effective refractive index" for 2D dark quasisolitons in an inhomogeneous BEC.

Then the equation of the quasisoliton trajectory takes the form typical of geometrical optics:

$$\frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}\tau} = \nu\big(g(s,0)\big)\mathbf{s}_0 = \mathbf{p}, \qquad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \frac{1}{2}\nabla\big(\nu^2\big)\Big|_{\eta=0}.$$

3. Method of calculation the trajectories of 2D dark quasisolitons in a smoothly inhomogeneous BEC

- First of all, we seek the density distribution of the BEC formed under the action of the external potential $V_{ext}(\mathbf{r})$ in the Thomas-Fermi approximation:

$$n_{g_{TF}}(\mathbf{r}) = g^2_{TF}(\mathbf{r}) = \big(1 - V_{ext}(\mathbf{r})\big) > 0.$$

- Then, using our analytical approximation for the dependence $\bar{\mathcal{P}}(\bar{\mathcal{E}})$, we determine the effective refractive index:

$$\nu(\mathbf{r}) \approx \nu_{TF}(\mathbf{r}) = \sqrt{n_{g_{TF}}(\mathbf{r})}\,\frac{\alpha_{TF}(\mathbf{r})}{\bar{\mathcal{E}}_0\, n_{0,g_{TF}}}\sinh\left(\frac{\bar{\mathcal{E}}_0\, n_{0,g_{TF}}}{\alpha_{TF}(\mathbf{r})\, n_{g_{TF}}(\mathbf{r})}\right),$$

$$\alpha_{TF}(\mathbf{r}) = 2\pi + \frac{2\pi}{3}\exp\left[-\left(\frac{\bar{\mathcal{E}}_0\, n_{0,g_{TF}}}{9.8\, n_{g_{TF}}(\mathbf{r})}\right)^2\right].$$

Here, $\bar{\mathcal{E}}_0$ and $n_{0,g_{TF}}$ are the normalized energy of the 2D dark quasisoliton and the density of the undisturbed inhomogeneous BEC at the point $\mathbf{r} = \mathbf{r}_0(0)$ from which the quasisoliton starts.

- Finally, we solve the geometrical-optics equation

$$\frac{\mathrm{d}^2\mathbf{r}_0}{\mathrm{d}\tau^2} = \frac{1}{2}\nabla\left(\nu_{_{TF}}^2\right)\Big|_{\mathbf{r}=\mathbf{r}_0}$$

with the "initial conditions"

$$\mathbf{r}_0(\tau=0) = \mathbf{r}_0(t=0), \qquad \frac{\mathrm{d}\mathbf{r}_0(\tau)}{\mathrm{d}\tau}\bigg|_{\tau=0} = \left[\nu_{_{TF}}(\mathbf{r}_0(t))\,\frac{\dot{\mathbf{r}}_0(t)}{|\dot{\mathbf{r}}_0(t)|}\right]\bigg|_{t=0}.$$

Here, $\mathbf{r}_0(t=0)$ and $\dot{\mathbf{r}}_0(t=0)$ are the position and velocity of a quasi-soliton at the time $t=0$ when we started to follow its motion.

Thus, we determine the trajectory $\mathbf{r}_0(\tau)$, along which a 2D dark quasisoliton moves with the energy

$$\mathcal{E}(s_{qs}) = g^2(s_{qs},0)\,\bar{\mathcal{E}}(s_{qs}) = g^2(0,0)\,\bar{\mathcal{E}}(0),$$

and the velocity

$$v(\mathbf{r}_0) = \sqrt{n_{g_{TF}}(\mathbf{r}_0)}\;\bar{v}\big(\bar{\mathcal{E}}(\mathbf{r}_0)\big).$$

V. Numerical simulation of the dynamics of 2D dark quasisolitons in a smoothly inhomogeneous BEC

The comparisons of the results of numerical simulation of the dynamics of 2D dark quasisolitons directly within GP equation with the results obtained by using the developed asymptotic theory confirmed the validity of the proposed method of describing the behavior of quasisolitons in a smoothly inhomogeneous BEC.

It was analysed in more detail

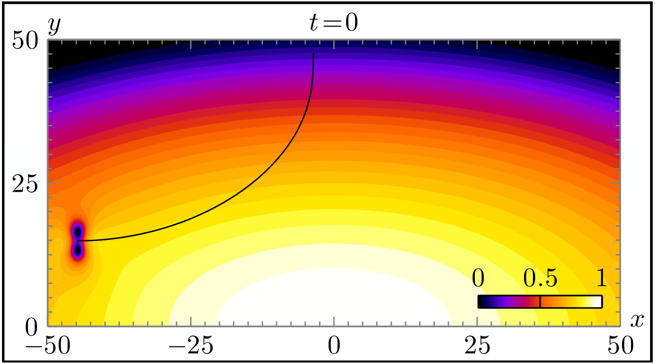
1. the trajectories, along which 2D dark quasisolitons move,
2. and the structural changes in the 2D dark quasisolitons moving along the corresponding propagation paths.

In our numerical calculations the wave function $\Psi(\mathbf{r},t)$ was specified at the initial time $t=0$ in the form of the product

$$\Psi(\mathbf{r},t=0) = g(\mathbf{r})\, F_0(\xi,\eta,v_0)\,.$$

Here, we calculated the purely real function $g(\mathbf{r})$ without the Thomas-Fermi approximation, but using the numerical solution of stationary equation. The wave function $F_0(\xi,\eta,v_0) \equiv \Psi_s\big(\bar{\xi},\bar{\eta},\bar{v}\big(\bar{\mathcal{E}}_0\big)\big)$ was also found by numerical solution of stationary problem.

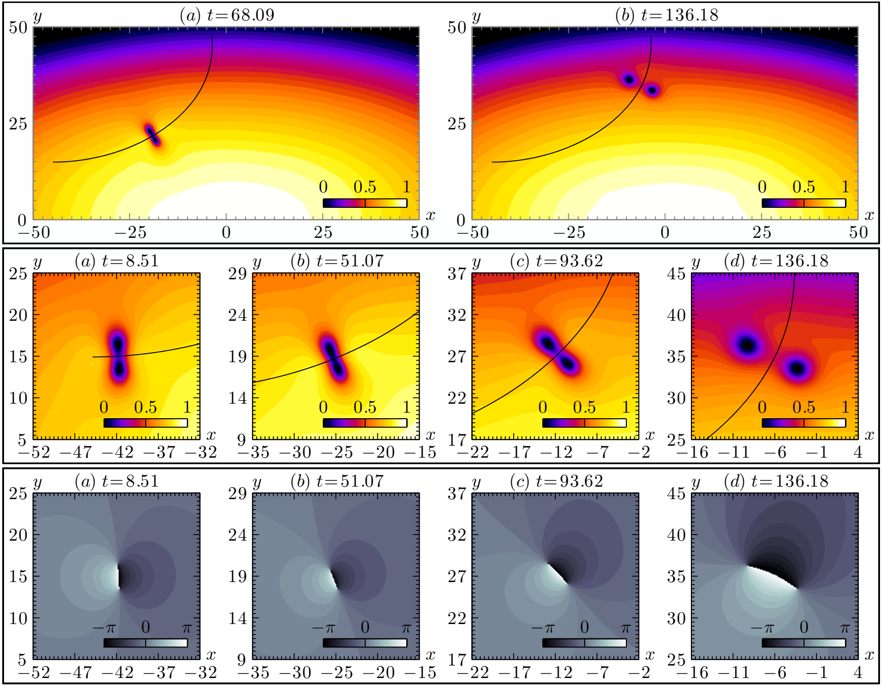
Sample 1.



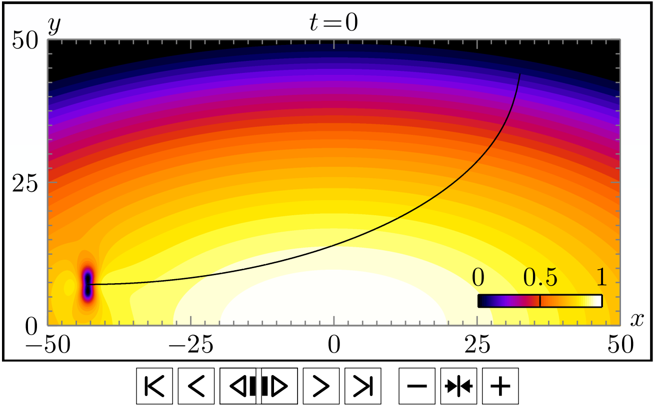
$$V_{ext}(\mathbf{r}) = \frac{\omega_x^2 x^2}{2} + \frac{\omega_y^2 y^2}{2}, \quad \omega_x = 0.0135, \quad \omega_y = 0.027.$$

$$x_0 = -44.82, \quad y_0 = -14.94, \quad n_{0,g_{TF}} = 0.734.$$

$$\bar{\mathcal{E}}_0 = 10.48, \quad \bar{v} = 0.4.$$


(a) t = 68.09
(b) t = 136.18
(a) t = 8.51
(b) t = 51.07
(c) t = 93.62
(d) t = 136.18
(a) t = 8.51
(b) t = 51.07
(c) t = 93.62
(d) t = 136.18
x
y
0 0.5 1
−π 0 π

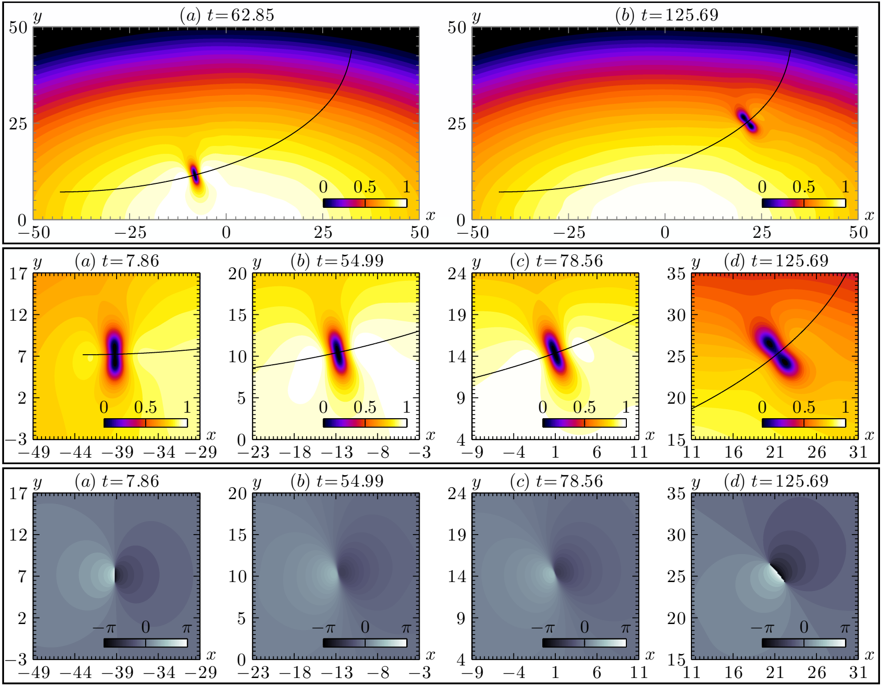
Sample 2.

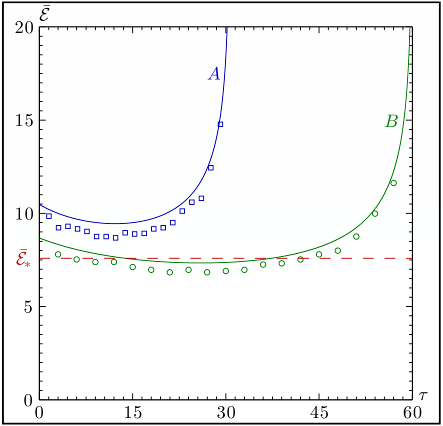


$$V_{ext}(\mathbf{r}) = \frac{\omega_x^2 x^2}{2} + \frac{\omega_y^2 y^2}{2}, \quad \omega_x = 0.014, \quad \omega_y = 0.028.$$

$$x_0 = -43.06, \quad y_0 = -7.18, \quad n_{0,g_{TF}} = 0.795.$$

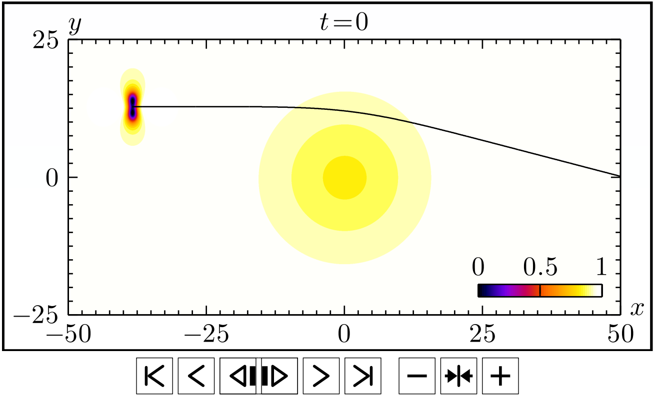
$$\bar{\mathcal{E}}_0 = 8.68, \quad \bar{v} = 0.525.$$


(a) t=62.85
(b) t=125.69
(a) t=7.86
(b) t=54.99
(c) t=78.56
(d) t=125.69
(a) t=7.86
(b) t=54.99
(c) t=78.56
(d) t=125.69



Variations in the normalized energy $\bar{\mathcal{E}}$ of a 2D dark quasisoliton along its propagation path $\mathbf{r}_0(\tau)$ in cases corresponding to the diagram A in the sample 1 and to the diagram B in the sample 2. The dashed line shows the critical values $\bar{\mathcal{E}} = \bar{\mathcal{E}}_* \approx 7.59$ of the normalized energy, at which the normalized velocity $\bar{v}$ of a 2D dark soliton is equal to $\bar{v}_* \approx 0.61$ and the soliton state undergoes a bifurcation (the vortex 2D dark soliton converts into a vortex-free one). The unshared squares and circles show the results obtained by numerical simulation performed directly within the framework of GP equation.

Sample 3.



$$V_{ext}(\mathbf{r}) = V_0 \exp\big(-\big(x^2+y^2\big)/\Lambda_G^2\big), \quad V_0 = 0.145, \quad \Lambda_G = 12.8.$$

$$x_0 = -38.4, \quad y_0 = 12.8, \quad n_{0,g_{TF}} \approx 1. \quad \bar{\mathcal{E}}_0 = 8.93, \quad \bar{v} = 0.5.$$

By artificially creating inhomogeneities in the BEC, using, e.g., focused laser beams, it is possible to control the behavior of 2D dark quasisolitons.
Our approach makes it possible to effectively select parameters of the laser beams.

VI. Conclusions

1. We have found the analytical approximation for the dependence of the momentum of 2D dark solitons on their energy.
2. We established that the trajectories of 2D dark quasisolitons in a smoothly inhomogeneous BEC are described by a system of ordinary differential equations, which we reduced to a form typical of geometrical optics of isotropic media. To this end, we introduced the concept of an effective refractive index dependent on both the density distribution of the undisturbed inhomogeneous condensate and the energy of the quasisoliton propagating in it.
3. We have found the law of variation in the normalized energy of 2D dark quasisolitons along their propagation paths. Using this law, one can describe the structural transformations of the quasisoliton formations moving in a smoothly inhomogeneous BEC.
4. We compared the results obtained by numerical simulation directly within the framework of the GP equation and the results of the analysis based on the constructed asymptotic description. This comparison shows good agreement between direct numerical calculations and the developed theory, which confirms its validity.