The Klein-Gordon Equation and Differential Substitutions

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Problem Statement

This report is devoted to the classification problem of nonlinear hyperbolic equations

$$u_{xy} = f(u, u_x, u_y). \tag{1}$$

The existence of higher symmetries is the hallmark of integrability of the equation. Zhiber and Shabat [1] obtained a complete list of nonlinear hyperbolic equations of the form

$$\boxed{v_{xy} = F(v)} \tag{2}$$

– the Klein-Gordon equations with higher symmetries.

The purpose of this paper is to describe all nonlinear hyperbolic equations of the form (1) reduced to Eq. (2) by differential substitutions

$$\boxed{v = \varphi(u, u_x, u_y).} \tag{3}$$

Note that we described the pairs of Eqs. (1), whose linearizations were related by Laplace transformations of the first and the second order. In this case we obtained differential substitutions which connected such nonlinear equations.

[1] Zhiber A. V., Shabat A. B., *Soviet Physics Doklady* **24** (1979), 607–609

[2] Kuznetsova M. N., *Ufimsk. Mat. Zh.* **1**:3 (2009), 87–96 (in Russian)

Theorem 1. *Suppose that Eq. (1) is reduced to Eq. (2) by differential substitution*

$$\boxed{v = \varphi(u, u_x).} \tag{4}$$

Then Eqs. (1), (2) and substitution (4) take one of the following form:

$$u_{xy} = uF'\big(F^{-1}(u_x)\big), \quad v_{xy} = F(v), \quad v = F^{-1}(u_x); \tag{5}$$

$$u_{xy} = \frac{\sqrt{2u_y}}{s'(u_x)}, \quad v_{xy} = F(v), \quad v = s(u_x), \quad s'(u_x)F\big(s(u_x)\big) = 1; \tag{6}$$

$$u_{xy} = u_x\big(\psi(u, u_y) - u_y\alpha'(u)\big), \quad v_{xy} = \exp v, \quad v = \alpha(u) + \ln u_x,$$
$$\psi_u + \psi\psi_{u_y} - \alpha' u_y \psi_{u_y} = \exp\alpha; \tag{7}$$

$$u_{xy} = u_x\big(\psi(u, u_y) - u_y\alpha'(u)\big), \quad v_{xy} = 0, \quad v = \alpha(u) + \ln u_x,$$
$$\psi_u + \psi\psi_{u_y} - \alpha' u_y \psi_{u_y} = 0; \tag{8}$$

$$u_{xy} = \frac{c - u_y\varphi_u(u, u_x)}{\varphi_{u_x}(u, u_x)}, \quad v_{xy} = 0, \quad v = \varphi(u, u_x); \tag{9}$$

$$u_{xy} = \sin u\sqrt{1-u_x^2}, \quad v_{xy} = \sin v, \quad v = u + \arcsin u_x; \tag{10}$$

$$u_{xy} = \exp u\sqrt{1+u_x^2}, \quad v_{xy} = \exp v, \quad v = u + \ln\left(u_x + \sqrt{1+u_x^2}\right); \tag{11}$$

$$u_{xy} = u, \quad v_{xy} = v, \quad v = cu + u_x; \tag{12}$$

$$u_{xy} = \delta(u_y), \quad v_{xy} = 1, \quad v = cu + u_x, \quad \delta(c+\delta') = 1 \tag{13}$$

up to point transformations $u \to \theta(u)$, $v \to \kappa(v)$, $x \to \alpha x$, $y \to \beta y$, where α, β are arbitrary constants. Here c is an arbitrary constant; in case (7) ψ, α satisfy the condition $\psi \neq u + u_y\alpha'$.

Now let us look at some of the obtained equations. The equation $u_{xy} = \sin u\sqrt{1-u_x^2}$ has symmetries of the third order [3]. Integrals and general solution of $u_{xy} = \exp u\sqrt{1+u_x^2}$ are contained in [3].

[3] Meshkov A.G., Sokolov V.V. *Theor. Math. Phys.* **166**:1 (2011), 43–57

Case (5)

If $F(v) = \exp v$, then we obtain

$$u_{xy} = uu_x. \tag{14}$$

Eq. (14) is reduced to the Liouville equation $v_{xy} = \exp v$ by differential substitution

$$v = \ln u_x. \tag{15}$$

Symmetries of the third order, integrals and general solution of Eq. (14) are presented in [3].

If $F(v) = \sin v$, then we obtain

$$u_{xy} = u\sqrt{1 - u_x^2}. \tag{16}$$

Eq. (16) is reduced to the sin-Gordon equation $v_{xy} = \sin v$ by differential substitution

$$v = \arcsin u_x. \tag{17}$$

Eq. (16) has symmetries of the third order [3].

[3] Meshkov A.G., Sokolov V.V. *Theor. Math. Phys.* **166**:1 (2011), 43–57

In case $F(v) = \exp v + \exp(-2v)$ we have

$$u_{xy} = 3ub(u_x). \tag{18}$$

Here b is determined by

$$(2u_x + b)^2(u_x - b) = 1. \tag{19}$$

Differential substitution

$$v = -\frac{1}{2}\ln\big(u_x - b(u_x)\big) \tag{20}$$

reduces Eq. (18) to the Tzitzeica equation $v_{xy} = \exp v + \exp(-2v)$ and is known (see. [4]).

[4] Zhiber A. V., Sokolov V. V. *Russian Mathematical Surveys* **56**(1):61 (2001), 61–101

Case (6)

If $F(v) = v$, then we obtain the Goursat equation

$$u_{xy} = 2\sqrt{u_x u_y}. \tag{21}$$

Eq. (21) is reduced to the equation

$$v_{xy} = v \tag{22}$$

by differential substitution

$$v = \sqrt{2u_x}. \tag{23}$$

Eq. (21) has symmetries of the third order [3].

If $F(v) = \sin v$ we get S–integrable equation (see [3])

$$u_{xy} = \sqrt{2u_y}\sqrt{1 - u_x^2}. \tag{24}$$

Eq. (24) is reduced to the sin-Gordon equation by differential substitution

$$v = \arccos(-u_x). \tag{25}$$

[3] Meshkov A.G., Sokolov V.V. *Theor. Math. Phys.* **166**:1 (2011), 43–57

If $F(v) = \exp v$, then we get

$$u_{xy} = u_x\sqrt{2u_y}. \tag{26}$$

Eq. (26) is reduced to the Liouville equation by substitution

$$v = \ln u_x. \tag{27}$$

Symmetries of the third order, integrals and general solution of Eq. (26) are presented in [3].

Of interest is the equation obtained in the case
$F(v) = \exp v + \exp(-2v)$:

$$u_{xy} = \sqrt{2u_y}a(u_x). \tag{28}$$

Here a is determined by

$$2(a + 2u_x)^2(a - u_x) = 27. \tag{29}$$

Differential substitution

$$v = -\frac{1}{2}\ln\left(\frac{2a(u_x) - 2u_x}{3}\right) \tag{30}$$

reduces Eq. (28) to the Tzitzeica equation. We can hope that Eq. (28) has simmetries of the fifth order.

[3] Meshkov A.G., Sokolov V.V. *Theor. Math. Phys.* **166**:1 (2011), 43–57

Case (9). Equation

$$u_{xy} = \frac{c - u_y \varphi_u(u, u_x)}{\varphi_{u_x}(u, u_x)} \tag{31}$$

in case $c = 0$ has x-integral $W = \varphi(u, u_x)$, in case $c \neq 0$ has x-integral $W = \varphi_{u_x} u_{xx} + \varphi_u u_x$.

Case (7). Equation

$$u_{xy} = u_x \big(c_1 \exp(-u) + c_2 \exp(u)\big). \tag{32}$$

is reduced to the Liouville equation $v_{xy} = 2c_2 \exp v$ by substitution

$$v = u + \ln u_x \tag{33}$$

Symmetries, integrals, and the general solution of Eq. (32) are presented in [3].

In case $\alpha(u) = u$ we obtain

$$u_{xy} = u_x \frac{\exp(u) - \psi_u(u, u_y)}{\psi_{u_y}(u, u_y)} \tag{34}$$

with y-integral $\bar{W} = \psi(u, u_y) - \exp u$.

[3] Meshkov A.G., Sokolov V.V. *Theor. Math. Phys.* **166**:1 (2011), 43–57

In general case Eq. (7)

$$u_{xy} = u_x\big(\psi(u, u_y) - u_y\alpha'(u)\big), \quad \text{where} \quad \psi_u + \psi\psi_{u_y} - \alpha' u_y \psi_{u_y} = \exp\alpha \tag{35}$$

has integrals

$$\bar{W} = \psi_{u_y} u_{yy} + \psi_u u_y - \frac{\psi^2}{2}, \tag{36}$$

$$W = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} + \left(\alpha''(u) - \frac{\alpha'^2(u)}{2}\right) u_x^2 \tag{37}$$

Case (8). Equation

$$u_{xy} = u_x\big(\psi(u, u_y) - u_y\alpha'(u)\big), \quad \text{where} \quad \psi_u + \psi\psi_{u_y} - \alpha' u_y \psi_{u_y} = 0 \tag{38}$$

has integrals

$$W = \frac{u_{xx}}{u_x} + \alpha'(u) u_x, \quad \bar{W} = \psi(u, u_y). \tag{39}$$

All the above equations with integrals are contained in a list of Liouville type equations given in [4].

[4] Zhiber A. V., Sokolov V. V. *Russian Mathematical Surveys* **56**(1):61 (2001), 61–101

Theorem 2. *Suppose that the Klein-Gordon equation*

$$\boxed{v_{xy} = F(v)} \tag{40}$$

is reduced to

$$\boxed{u_{xy} = f(u, u_x, u_y)} \tag{41}$$

by differential substitution

$$\boxed{u = \psi(v, v_y)} \tag{42}$$

Then Eqs. (40), (41) *and substitution* (42) *take one of the following form:*

$$v_{xy} = F(v), \quad u_{xy} = F'\big(F^{-1}(u_x)\big)u, \quad u = v_y; \tag{43}$$

$$v_{xy} = 1, \quad u_{xy} = \frac{\psi''\big(\psi^{-1}(u)\big)u_y}{\psi'\big(\psi^{-1}(u)\big)}, \quad u = \psi(v_y); \tag{44}$$

$$v_{xy} = 0, \quad u_{xy} = 0, \quad u = cv + \mu(v_y); \tag{45}$$

$$v_{xy} = 0, \quad u_{xy} = -u_x \exp u, \quad u = \ln v_y - \ln v; \tag{46}$$

$$v_{xy} = v, \quad u_{xy} = u, \quad u = cv + v_y; \tag{47}$$

$$v_{xy} = 1, \quad u_{xy} = 1, \quad u = v + v_y \tag{48}$$

up to point transformations $v \to \kappa(v)$, $u \to \theta(u)$, $x \to \alpha x$, $y \to \beta y$, *where* α *and* β *– are arbitrary constants. Here* c *– is an arbitrary constant.*

Based on the above lists, the Bäcklund transformations have been constructed for some pairs of equations. Namely, equations

$$u_{xy} = -u_x \exp u, \quad v_{xy} = 0 \tag{49}$$

are connected by the Bäcklund transformation

$$v = \ln u_x - u, \quad u = \ln v_y - \ln v. \tag{50}$$

Equations

$$u_{xy} = F'\big(F^{-1}(u_x)\big)u, \quad v_{xy} = F(v) \tag{51}$$

are related by the Bäcklund transformation

$$v = F^{-1}(u_x), \quad u = v_y. \tag{52}$$

We showed (see [2]) that linearizations of Eqs. (51) were related by the Laplace transformation of the first order.

[2] Kuznetsova M. N. *Ufimsk. Mat. Zh.* **1**:3 (2009), 87–96 (in Russian)

The purpose of this section is the classification of nonlinear hyperbolic equations

$$\boxed{u_{xy} = f(u, u_x, u_y),} \tag{53}$$

reduced to the Klein-Gordon equation

$$\boxed{v_{xy} = F(v)} \tag{54}$$

by differential substitution of the form

$$\boxed{v = \varphi(u, u_x, u_y), \quad \varphi_{u_x} \cdot \varphi_{u_y} \neq 0.} \tag{55}$$

Theorem 3. *Suppose that Eq.* (53) *is reduced to Eq.* (54) *by differential substitution* (55) . *Then Eqs.* (53), (54) *and substitution* (55) *take one of the following form:*

$$u_{xy} = \sqrt{u_x^2 + a}\sqrt{u_y^2 + b}, \quad v_{xy} = \frac{1}{2}\big(\exp v - ab\exp(-v)\big),$$

$$v = \ln\left(u_x + \sqrt{u_x^2 + a}\right)\left(u_y + \sqrt{u_y^2 + b}\right); \tag{56}$$

$$u_{xy} = \sqrt{u_x u_y}, \quad v_{xy} = v/4, \quad v = \sqrt{u_x} + \sqrt{u_y}; \tag{57}$$

$$u_{xy} = \sqrt{u_x}, \quad v_{xy} = 1/2, \quad v = \sqrt{u_x} + u_y; \tag{58}$$

$$u_{xy} = 1, \quad v_{xy} = 0, \quad v = u_x + u_y; \tag{59}$$

$$u_{xy} = \frac{1}{\gamma'(u_y)}, \quad v_{xy} = 1, \quad v = u_x + \gamma(u_y) + u, \quad 1 - \frac{\gamma''}{\gamma'^2} = \gamma'; \tag{60}$$

$$u_{xy} = \mu(u) u_x u_y, \quad v_{xy} = 0, \quad v = c_1 \ln u_x + c_2 \ln u_y + \alpha(u),$$
$$\mu'(c_1 + c_2) + \mu^2(c_1 + c_2) + \alpha'' + \alpha'\mu = 0; \tag{61}$$

$$u_{xy} = \mu(u) u_x u_y, \quad v_{xy} = \exp v, \quad v = \ln(u_x u_y) + \alpha(u),$$
$$2\mu' + 2\mu^2 + \alpha'' + \alpha'\mu = \exp\alpha; \tag{62}$$

$$u_{xy} = u, \quad v_{xy} = v, \quad v = c_1 u_y + c_2 u_x + c_3 u; \tag{63}$$

$$u_{xy} = \mu(u)(u_y + c)u_x, \quad v_{xy} = \exp v, \quad v = \ln(u_y + c) + \ln u_x + \alpha(u),$$
$$2\mu' + 2\mu^2 + \alpha'' + \alpha'\mu = \exp\alpha, \quad 2\mu^2 + \mu' + \alpha'\mu = \exp\alpha; \tag{64}$$

$$u_{xy} = \mu(u)(u_y + c)u_x, \quad v_{xy} = 0, \quad v = c_2 \ln(u_y + c) + c_1 \ln u_x + \alpha(u),$$
$$(\mu' + \mu^2)(c_1 + c_2) + \alpha'' + \alpha'\mu = 0, \quad c_1\mu' + \mu^2(c_1 + c_2) + \alpha'\mu = 0; \tag{65}$$

$$u_{xy} = \mu(u) u_x, \quad v_{xy} = 0, \quad v = u_y - \ln u_x + \alpha(u),$$
$$\alpha'' + \mu' = 0, \quad \mu^2 - \mu' + \alpha'\mu = 0; \tag{66}$$

$$u_{xy} = \frac{\mu(u) u_x}{\gamma'(u_y)}, \quad v_{xy} = 0, \quad v = \ln u_x + \gamma(u_y) + \alpha(u), \tag{67}$$

$$c_3 + \frac{\gamma''}{\gamma'^2} + c_4\gamma' u_y = 0, \alpha'' + \mu' + c_4\mu^2 = 0, c_3\mu^2 + \mu' + \mu^2 + \alpha'\mu = 0;$$

$$u_{xy} = 0, \quad v_{xy} = 0, \quad v = \beta(u_x) + \gamma(u_y) + c_3 u; \tag{68}$$

$$u_{xy} = \frac{u_x}{(au+b)\gamma'(u_y)}, \quad v_{xy} = \exp v, \quad v = \ln u_x + \gamma(u_y) - 2\ln(au+b),$$

$$c_3 + \frac{\gamma''}{\gamma'^2} + c_4\gamma' u_y = -\gamma' \exp\gamma, c_3 + 1 - 3a = 0, c_4 + 2a^2 - a = 0; \tag{69}$$

$$u_{xy} = -\frac{1}{u\beta'(u_x)\gamma'(u_y)}, \quad v_{xy} = 0, \quad v = \beta(u_x) + \gamma(u_y),$$

$$\frac{\beta''}{\beta'^2} = u_x\beta' + c_1, \quad \frac{\gamma''}{\gamma'^2} = u_y\gamma' - c_1; \tag{70}$$

$$u_{xy} = \frac{\mu(u)}{\beta'(u_x)\gamma'(u_y)}, \quad v_{xy} = \exp v, \quad v = \beta(u_x) + \gamma(u_y) + \alpha(u),$$

$$u_x + \frac{1}{\beta'(u_x)} = \exp(\beta), u_y + \frac{1}{\gamma'(u_y)} = \exp\gamma, \alpha'' = \exp\alpha, \mu = (\exp\alpha)/\alpha'; \tag{71}$$

$$u_{xy} = \frac{\mu(u)}{\beta'(u_x)\gamma'(u_y)}, \quad v_{xy} = \exp v, \quad v = \beta(u_x) + \gamma(u_y) + \alpha(u),$$

$$2u_x + \frac{1}{\beta'(u_x)} = \exp\beta, \quad 2u_y + \frac{1}{\gamma'(u_y)} = \exp\gamma,$$

$$\alpha'\mu - 2\mu^2 = \exp\alpha, \quad \alpha'^2 = 8\exp\alpha; \tag{72}$$

$$u_{xy}=s(u)\sqrt{1-u_x^2}\sqrt{1-u_y^2},\quad v_{xy}=c\sin v,$$
$$v=\arcsin u_x+\arcsin u_y+p(u),$$
$$s''-2s^3+\lambda s=0,\quad p'^2=2s'-2s^2+\lambda; \tag{73}$$
$$u_{xy}=s(u)b(u_x)\bar{b}(u_y),\quad v_{xy}=c_1\exp v+c_2\exp(-2v),$$
$$v=-\frac{1}{2}\ln(u_x-b)-\frac{1}{2}\ln(u_y-\bar{b})+p(u),$$
$$(u_x-b)(b+2u_x)^2=1,\quad (u_y-\bar{b})(\bar{b}+2u_y)^2=1,$$
$$s''-2ss'-4s^3=0,\quad p'^2-2sp'-3s'-2s^2=0 \tag{74}$$

up to point transformations $u\to\theta(u)$, $v\to\kappa(v)$, $x\to\alpha x$, $y\to\beta y$ *and change of the form* $u+\alpha x+\beta y\to u$, *where* α, β *– are arbitrary constants. Here* a, b, c, c_1, c_2, c_3, c_4 *are arbitrary constants such that* $a^2+b^2\neq 0$, $c\cdot c_1\cdot c_2\neq 0$; *in cases* (67), (69) *the function* γ *satisfies the condition* $\left(\gamma''/\gamma'^2\right)'\neq 0$; *in cases* (70)–(72) *the functions* β *and* γ *sutisfy the conditions* $\left(\beta''/\beta'^2\right)'\neq 0$ *and* $\left(\gamma''/\gamma'^2\right)'\neq 0$ *accordingly, and* $\mu'\neq 0$; *everywhere* $\mu\neq 0$.

Consider equation

$$u_{xy} = \sqrt{u_x^2 + a}\sqrt{u_y^2 + b}. \tag{75}$$

If $a \cdot b \neq 0$, then, after transformations $\sqrt{a}x \to x$, $\sqrt{b}y \to y$, $v - \ln(ab)^{1/2} \to v$, we obtain

$$u_{xy} = \sqrt{u_x^2 + 1}\sqrt{u_y^2 + 1}. \tag{76}$$

Eq. (76) reduced to the sin-Gordon equation

$$v_{xy} = \frac{1}{2}\Big(\exp v + \exp(-v)\Big) \tag{77}$$

by differential substitution

$$v = \ln\Big(u_x + \sqrt{u_x^2 + 1}\Big)\Big(u_y + \sqrt{u_y^2 + 1}\Big). \tag{78}$$

Note that Eqs. (76)–(78) is reduced to

$$u_{xy} = \sqrt{1 - u_x^2}\sqrt{1 - u_y^2}, v = \arcsin u_x + \arcsin u_y, v_{xy} = -\sin v \tag{79}$$

by point transformations and transformations of independent variables. Eq. (76) is S-integrable and have symmetries of the third order (see, [3]).

If $a = 0$ then transformations $v - \ln 2 \to v$, $\sqrt{b}y \to y$, $v - \ln\sqrt{b} \to v$ give

$$u_{xy} = u_x\sqrt{u_y^2 + 1}. \tag{80}$$

Eq. (80) is reduced to the Liouville equation

$$v_{xy} = \exp v \tag{81}$$

by differential substitution

$$v = \ln u_x + \ln\Big(u_y + \sqrt{u_y^2 + 1}\Big). \tag{82}$$

Eqs. (80)–(82) are reduced to

$$u_{xy} = u_x\sqrt{1 - u_y^2}, \quad v = -i \arcsin u_y + \ln u_x, \quad v_{xy} = -i \exp v \tag{83}$$

by point transformations and transformations of independent variables. Symmetries of the third order, x and y-integrals, and general solution of Eq. (80) are contained in [3].

Equation
$$u_{xy} = \sqrt{u_x} \tag{84}$$
has symmetries of the third order [3]. The integrals of this equation are given by the formulas
$$\omega = \frac{u_{xx}}{\sqrt{u_x}}, \quad \bar{\omega} = u_{yyy}. \tag{85}$$

Equation
$$u_{xy} = \mu(u) u_x u_y \tag{86}$$
has integrals $\omega = \ln u_x - \sigma(u)$, $\bar{\omega} = \ln u_y - \sigma(u)$. Here $\sigma' = \mu$.

Equation
$$u_{xy} = \mu(u)(u_y + c) u_x \tag{87}$$
in case (64) has the y-integral of the first order
$$\bar{\omega} = \ln(u_y + c) - \sigma(u), \tag{88}$$
where $\sigma' = \mu$, $c \neq 0$. At the same time the x-integral has the form
$$\omega = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} - \frac{1}{2}\big(\mu^2(u) + 2\mu(u)\alpha'(u) + \alpha'^2(u)\big)u_x^2. \tag{89}$$
In case (14) the x-integral has the form $\omega = c_2 \mu(u) u_x + c_1 \frac{u_{xx}}{u_x} + \alpha'(u) u_x$.

Equation

$$u_{xy} = \frac{u_x}{(au+b)\gamma'(u_y)} \tag{90}$$

has the y-integral of the first order

$$\bar{\omega} = \gamma(u_y) - \frac{1}{a}\ln(au+b) \tag{91}$$

and the x-integral of the third order

$$\omega = \frac{u_{xxx}}{u_x} - \frac{3}{2}\frac{u_{xx}^2}{u_x^2} + \frac{u_x^2(2a-1)}{2(au+b)^2}. \tag{92}$$

The equation

$$u_{xy} = \frac{\mu(u)}{\beta'(u_x)\gamma'(u_y)} \tag{93}$$

under conditions (71) is reduced to equation from the list of paper [4] by point transformation. Eq. (93) has the integrals of the second order

$$\omega = \beta'(u_x)u_{xx} - \frac{\mu'(u)}{\mu(u)\beta'(u_x)}, \quad \bar{\omega} = \gamma'(u_y)u_{yy} - \frac{\mu'(u)}{\mu(u)\gamma'(u_y)}. \tag{94}$$

Eq. (93) under conditions (72) is reduced to equation presented in [4]

$$u_{xy} = \frac{1}{u} B(u_x) \bar{B}(u_y) \tag{95}$$

by point transformation. Here $BB' + B - 2u_x = 0$, $\bar{B}\bar{B}' + \bar{B} - 2u_y = 0$. There are integrals of Eq. (95) in the above paper

$$\omega = \frac{u_{xxx}}{B} + \frac{2(B - u_x)}{B^3} u_{xx}^2 + \frac{2(2u_x + B)}{uB} + \frac{B(u_x + B)}{u^2},$$
$$\bar{\omega} = \frac{u_{yyy}}{\bar{B}} + \frac{2(\bar{B} - u_y)}{\bar{B}^3} u_{yy}^2 + \frac{2(2u_y + \bar{B})}{u\bar{B}} + \frac{\bar{B}(u_y + \bar{B})}{u^2}.$$

All of the above equations with integrals are contained in a list of Liouville type equations given in review [4].

[4] Zhiber A. V., Sokolov V. V. *Russian Mathematical Surveys* **56**(1):61 (2001), 61–101

Theorem 4. *Suppose that the Klein-Gordon equation*

$$\boxed{v_{xy} = F(v)} \tag{96}$$

is reduced to

$$\boxed{u_{xy} = f(u, u_x, u_y)} \tag{97}$$

by differential substitution of the form

$$\boxed{u = \psi(v, v_x, v_y), \quad \psi_{v_x} \cdot \psi_{v_y} \neq 0} \tag{98}$$

Then Eqs. (96), (97) *and substitution* (98) *take one of the following form:*

$$v_{xy} = v, \quad u_{xy} = u, \quad u = c_1 u_x + c_2 u_y + c_3 u; \tag{99}$$

$$v_{xy} = 0, \quad u_{xy} = 0, \quad u = \beta(v_x) + \gamma(v_y) + c_3 v; \tag{100}$$

$$v_{xy} = 0, u_{xy} = \exp(u) u_y, p' = \exp(cv), u = \ln\left(-\frac{p'(v) v_x}{\mu(v_y) + p(v)}\right); \tag{101}$$

$$v_{xy} = 1, \quad u_{xy} = c_1(u_x - c_2), \quad u = \exp(c_1 v_x) + c_2 v_y; \tag{102}$$

$$v_{xy} = \exp v, \quad u_{xy} = u u_x, \quad u = v_y + \mu(v_x) \exp v, \quad 2\mu' = \mu^2; \tag{103}$$

$$v_{xy} = 0, u_{xy} = \exp u, u = \ln v_x + \ln v_y + \delta(v), \delta''(v) = \exp \delta(v); \tag{104}$$

$$v_{xy} = 1, u_{xy} = c_1 u_x + c_2 u_y - c_1 c_2 u, u = \exp(c_1 v_x) + \exp(c_2 v_y) \tag{105}$$

up to point transformations $u \to \theta(u)$, $v \to \kappa(v)$, $x \to \alpha x$, $y \to \beta y$, *where* α, β *are arbitrary constants. Here* c, c_1, c_2 *are arbitrary constants such that* $c_1 \cdot c_2 \neq 0$.

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