Generalized symmetry classification of discrete equations in a class depending on 12 parameters

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Comparing quad graph discrete equations and hyperbolic equations

$u_{xy} = f(u, u_x, u_y)$ $D_y = \frac{d}{dy}$, $D_x = \frac{d}{dx}$	$u_{n+1,m+1} = f(u_{n,m}, u_{n+1,m}, u_{n,m+1})$ $T_1 h(n,m) = h(n+1,m)$, $T_2 h(n,m) = h(n,m+1)$
Darboux integrable equations $u_{xy} = e^u$ $W_1 = u_{xx} - u_x^2/2,\ D_y W_1 = 0$ $W_2 = u_{yy} - u_y^2/2,\ D_x W_2 = 0$	$u_{n+1,m+1}u_{n,m} - u_{n+1,m}u_{n,m+1} = 1$ $W_1 = \frac{u_{n+1,m}+u_{n-1,m}}{u_{n,m}},\ (T_2 - 1)W_1 = 0$ $W_2 = \frac{u_{n,m+1}+u_{n,m-1}}{u_{n,m}},\ (T_1 - 1)W_2 = 0$
Sin-Gordon type equations $u_{xy} = \sin u$ $u_t = u_{xxx} + u_x^3/2$ $u_\tau = u_{yyy} + u_y^3/2$	$u_{n+1,m+1}u_{n,m+1} + u_{n+1,m}u_{n,m}$ $+u_{n,m}u_{n,m+1} = 0$ $u_{n,m,t} = \frac{u_{n+1,m}u_{n,m}}{u_{n-1,m}}$ $u_{n,m,\tau} = \frac{u_{n,m}u_{n,m-1}}{u_{n,m+1}-u_{n,m-1}}$

Consider nonlinear equations on a quad graph (or double discrete chains) of the form

$$F(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0, \tag{1}$$

In (1) F is **polylinear** function (F - linear function of each of its variable). All integrable discrete equation known in the literature belong to this class. This class depends on 16 arbitrary constants. We restrict ourselves to the case when

$$\frac{\partial^2 F}{\partial u_{n+1,m+1} \partial u_{n+1,m}} = 0.$$

It depends on 12 parameters and can be written in the form

$$\begin{aligned} A u_{n+1,m+1} + B u_{n+1,m} + C = 0, \\ A = a_1 u_{n,m} u_{n,m+1} + a_2 u_{n,m} + a_3 u_{n,m+1} + a_4, \\ B = b_1 u_{n,m} u_{n,m+1} + b_2 u_{n,m} + b_3 u_{n,m+1} + b_4, \\ C = c_1 u_{n,m} u_{n,m+1} + c_2 u_{n,m} + c_3 u_{n,m+1} + c_4. \end{aligned} \tag{2}$$

Structure of symmetry

We use as integrability criterion the existence of a non-autonomous generalized symmetry of the form:

$$u_{n,m,t} = G_{n,m}(u_{n+1,m}, u_{n-1,m}, u_{n,m}, u_{n,m+1}, u_{n,m-1}). \qquad (3)$$

We consider a symmetry of the form (3) satisfying conditions:

$$\frac{\partial G_{n,m}}{\partial u_{n+1,m}}, \frac{\partial G_{n,m}}{\partial u_{n-1,m}}, \frac{\partial G_{n,m}}{\partial u_{n,m+1}}, \frac{\partial G_{n,m}}{\partial u_{n,m-1}} \neq 0, \qquad (4)$$

for all $n, m \in \mathcal{Z}$. It can be proved that the function G has the form:

$$G_{n,m} = \Phi_{n,m}(u_{n+1,m}, u_{n,m}, u_{n-1,m}) + \Psi_{n,m}(u_{n,m+1}, u_{n,m}, u_{n,m-1}). \qquad (5)$$

In practice we always obtain two generalized symmetry in the form:

$$u_{n,m,t_1} = \Phi_{n,m}, \quad u_{n,m,t_2} = \Psi_{n,m}. \qquad (6)$$

First integrals

In our work, analogues of the Liuville equation $u_{xy} = e^u$ appear in a natural way as equations which symmetries depend on arbitrary functions. Such equations are Darboux integrable, as they have two first integrals:

$$(T_1 - 1)W_2 = 0, \quad W_2 = w_n^{(2)}(u_{n,m+l_2}, u_{n,m+l_2+1}, \dots, u_{n,m+k_2}), \tag{7}$$

$$(T_2 - 1)W_1 = 0, \quad W_1 = w_m^{(1)}(u_{n+l_1,m}, u_{n+l_1+1,m}, \dots, u_{n+k_1,m}). \tag{8}$$

Here l_1, l_2, k_1, k_2 are integer numbers such that $l_1 < k_1, l_2 < k_2$; T_1, T_2 are shift operators in the first and second direction, respectively: $T_1 h_{n,m} = h_{n+1,m}$, $T_2 h_{n,m} = h_{n,m+1}$.

Nondegeneracy and Nonlinearity

We rewrite the (1) in the form

$$u_{n+1,m+1} = f(u_{n+1,m}, u_{n,m}, u_{n,m+1}) \qquad (9)$$

and require the essential dependence of f on all its variables. So the nondegeneracy conditions read:

$$\frac{\partial F}{\partial u_{n+1,m+1}}, \frac{\partial f}{\partial u_{n+1,m}}, \frac{\partial f}{\partial u_{n,m}}, \frac{\partial f}{\partial u_{n,m+1}} \neq 0. \qquad (10)$$

In our classification we reject the linear equations and equations equivalent to linear ones. The following equations are transformed into linear ones by point transform $u_{n,m} = e^{v_{n,m}}$

$$u_{n+1,m+1} = \nu u_{n+1,m} u_{n,m}^{\pm 1} u_{n,m+1}^{\pm 1}. \qquad (11)$$

We reject equations related to (11) by autonomous Möbius (linear-fractional) transformation

$$\tilde{u}_{n,m} = \frac{\alpha u_{n,m} + \beta}{\gamma u_{n,m} + \delta} \qquad (12)$$

Method of classification.

Here will use four integrability conditions which have the form of conservation laws:

$$(T_1 - 1)p^{(j)}_{n,m} = (T_2 - 1)q^{(j)}_{n,m}, \quad j = 1, 2, 3, 4. \tag{13}$$

The following statement takes place:

Theorem

If an equations (9) *has generalized symmetry of the form (3,4) then it must have conservation laws* (13) *such that:*

$$\begin{aligned}
p^{(1)}_{n,m} &= \log f_{u_{n+1,m}}, & q^{(1)}_{n,m} &= q^{(1)}_{n,m}(u_{n+2,m}, u_{n+1,m}, u_{n,m});\\
p^{(2)}_{n,m} &= \log \frac{f_{u_{n,m}}}{f_{u_{n,m+1}}}, & q^{(2)}_{n,m} &= q^{(2)}_{n,m}(u_{n+2,m}, u_{n+1,m}, u_{n,m});\\
q^{(3)}_{n,m} &= \log f_{u_{n,m+1}}, & p^{(3)}_{n,m} &= p^{(3)}_{n,m}(u_{n,m+2}, u_{n,m+1}, u_{n,m});\\
q^{(4)}_{n,m} &= \log \frac{f_{u_{n,m}}}{f_{u_{n+1,m}}}, & p^{(4)}_{n,m} &= p^{(4)}_{n,m}(u_{n,m+2}, u_{n,m+1}, u_{n,m}),
\end{aligned} \tag{14}$$

where $f_{u_{n+k,m+l}} = \frac{\partial f}{\partial u_{n+k,m+l}}$.

The annihilators are defined as:

$$
\begin{aligned}
Y_l = T_2^{-l}\frac{\partial}{\partial u_{n,m+1}}T_2^l, \quad Y_{-l} = T_2^l\frac{\partial}{\partial u_{n,m-1}}T_2^{-l}, \quad l>0,\\
Z_k = T_1^{-k}\frac{\partial}{\partial u_{n+1,m}}T_1^k, \quad Z_{-k} = T_1^k\frac{\partial}{\partial u_{n-1,m}}T_1^{-k}, \quad k>0.
\end{aligned} \tag{15}
$$

These operators are differentiation operators and their action will be clarified by example of Y_1.
It is obviously that:

$$
Y_1 u_{n,m+l} = 0, l \neq 0, \quad Y_1 u_{n,m} = 1.
$$

On other independent variables defined at fix point (n,m) it acts as follows:

$$
Y_1 u_{n+k,m} = \prod_{j=0}^{k-1} T_2^{-1} T_1^j \frac{\partial f^{(1,1)}}{\partial u_{n,m+1}}, \quad k>0,
$$

$$
Y_1 u_{n+k,m} = \prod_{j=k+1}^{0} T_2^{-1} T_1^j \frac{\partial f^{(-1,1)}}{\partial u_{n,m+1}}, \quad k<0.
$$

Lemma

The operators (15) *annihilate the following functions:*

$$Y_{-l}q^{(j)}_{n+k,m+l} = 0,\ \ l \neq 0,\ \ j = 1,2,$$
$$Z_{-k}p^{(j)}_{n+k,m+l} = 0,\ \ k \neq 0,\ \ j = 3,4.$$

The use of annihilators (15)**.** Applying operators:

$$\Lambda_{2,l} = \begin{cases} T_1^{-1}\sum_{j=0}^{l-1} T_2^j, & l > 0, \\ -T_1^{-1}\sum_{j=l}^{-1} T_2^j, & l < 0, \end{cases} \qquad \Lambda_{1,k} = \begin{cases} T_2^{-1}\sum_{j=0}^{k-1} T_1^j, & k > 0, \\ -T_2^{-1}\sum_{j=k}^{-1} T_1^j, & k < 0, \end{cases}$$

to integrability conditions (13) we get

$$\begin{aligned} \Lambda_{2,l}(T_1 - 1)p^{(j)}_{n,m} &= (T_2^l - 1)q^{(j)}_{n-1,m}, \quad l \neq 0,\ j = 1,2. \\ (T_1^k - 1)p^{(j)}_{n,m-1} &= \Lambda_{1,k}(T_2 - 1)q^{(j)}_{n,m}, \quad k \neq 0,\ j = 3,4. \end{aligned} \tag{16}$$

Applying $Y_{-l},\ Z_{-k}$ to the equations (16) respectively and taking into account Lemma 2, we get:

$$Y_{-l} q^{(j)}_{n-1,m} = r^{(l,j)}_{n,m}, \quad l \neq 0,\ j = 1, 2, \tag{17}$$

$$Z_{-k} p^{(j)}_{n,m-1} = s^{(k,j)}_{n,m}, \quad k \neq 0,\ j = 3, 4. \tag{18}$$

Here functions $r^{(l,j)}_{n,m}, s^{(k,j)}_{n,m}$ as well as coefficients of Y_{-l}, Z_{-k} are explicitly expressed in terms of eq. (2).
We have an infinite system of linear non-homogeneous PDEs for each of four functions $q^{(j)}_{n-1,m},\ p^{(j)}_{n,m-1}$. We can add to eqs. (17,18) linear PDEs with the commutators $[Y_{l_1}, Y_{l_2}]$ and $[Z_{k_1}, Z_{k_2}]$ instead of the operators Y_{-l} and Z_{-k}. Also we can decompose each equation with respect to additional variables $u_{n,m+j}$ in the case of (17) and with respect to $u_{n+j,m}$ in the case of (18).

Main Theorem

List

1 (Properties of equations used in classification)

1. *Nondegenerate, i.e. satisfying conditions* (10)*;*
2. *Possessing symmetries of the form (3,4);*
3. *Equations which do not have N-point first integrals with* $2 \leq N \leq 5$*;*
4. *Nonlinear equations and equations not equivalent to* (11)*.*

Theorem

An equation (2) *satisfies conditions of List 1 if and only if it is equivalent to an equation of List 2 up to a linear transform* $\hat{u}_{n,m} = \alpha u_{n,m} + \beta$*.*

List

2 (Equations (2) of the sine-Gordon type and of the Burgers type together with their symmetries of the form (6))

1.

$$(u_{n+1,m+1}-1)(u_{n,m+1}+1)=(u_{n+1,m}+1)(u_{n,m}-1),$$

$$\frac{d}{dt_1}u_{n,m}=(u_{n,m}^2-1)(u_{n+1,m}-u_{n-1,m}),$$

$$\frac{d}{dt_2}u_{n,m}=(u_{n,m}^2-1)\left(\frac{1}{u_{n,m+1}+u_{n,m}}-\frac{1}{u_{n,m}+u_{n,m-1}}\right)$$

2.

$$(u_{n+1,m+1}-u_{n+1,m}+c_2)(u_{n,m}-u_{n,m+1}-c_2)+u_{n+1,m}-u_{n,m+1}+c_4=0,$$

$$\frac{d}{dt_1}u_{n,m}=(u_{n+1,m}-u_{n,m}+c_2+c_4)(u_{n,m}-u_{n-1,m}+c_2+c_4),$$

$$\frac{d}{dt_2}u_{n,m}=\frac{(u_{n,m+1}-u_{n,m})(u_{n,m}-u_{n,m-1})-c_2-c_2^2}{u_{n,m+1}-u_{n,m-1}+2c_2+1}$$

3.

$$u_{n+1,m+1}u_{n,m+1}+b_2u_{n+1,m}u_{n,m}+c_1u_{n,m}u_{n,m+1}=0,\quad b_2,c_1\neq 0,$$

$$\frac{d}{dt_1}u_{n,m}=\frac{u_{n+1,m}u_{n,m}}{u_{n-1,m}},\qquad \frac{d}{dt_2}u_{n,m}=\frac{u_{n,m}u_{n,m-1}}{u_{n,m+1}-b_2u_{n,m-1}}$$

4.

$$u_{n+1,m+1}u_{n,m}+u_{n+1,m}u_{n,m+1}+a_3u_{n+1,m+1}u_{n,m+1}+b_2u_{n+1,m}u_{n,m}$$
$$+c_1u_{n,m}u_{n,m+1}=0,\quad c_1,b_2\neq 0,$$

$$\frac{d}{dt_1}u_{n,m}=(1-a_3b_2)\frac{u_{n+1,m}u_{n,m}}{u_{n-1,m}}+c_1\left(u_{n+1,m}+\frac{u_{n,m}^2}{u_{n-1,m}}\right),$$

$$\frac{d}{dt_2}u_{n,m}=\frac{(u_{n,m+1}+b_2u_{n,m})(u_{n,m}+b_2u_{n,m-1})}{a_3u_{n,m+1}-b_2u_{n,m-1}}$$

5.

$$u_{n+1,m+1}u_{n,m} + u_{n+1,m}u_{n,m+1} + a_3 u_{n+1,m+1}u_{n,m+1} + c_1 u_{n,m}u_{n,m+1} = 0, \quad c_1, a_3 \neq 0,$$

$$\frac{d}{dt_1}u_{n,m} = \frac{u_{n+1,m}u_{n,m}}{u_{n-1,m}} + c_1\left(u_{n+1,m} + \frac{u_{n,m}^2}{u_{n-1,m}}\right),$$

$$\frac{d}{dt_2}u_{n,m} = \frac{u_{n,m-1}u_{n,m}}{u_{n,m+1}} + a_3\left(u_{n,m-1} + \frac{u_{n,m}^2}{u_{n,m+1}}\right)$$

6. Discrete Burgers equation: $a_3, b_1, d, |d + b_1 a_3| + |b_1^2 - 1| \neq 0$

$$(u_{n+1,m+1} + d)(u_{n,m} + a_3)u_{n,m+1} + b_1(u_{n,m+1} + d)(u_{n+1,m} + a_3)u_{n,m} = 0,$$

$$\frac{d}{dt_1}u_{n,m} = (-b_1)^{-m}u_{n,m}(u_{n+1,m} - u_{n,m})C_1 + (-b_1)^m \frac{u_{n-1,m} - u_{n,m}}{u_{n-1,m}}C_2 \tag{19}$$

$$\frac{d}{dt_2}u_{n,m} = (-b_1)^n \frac{(u_{n,m} - u_{n,m+1})(u_{n,m} + a_3)}{u_{n,m+1} + d}C_3 + (-b_1)^{-n}\frac{(u_{n,m} - u_{n,m-1})(u_{n,m} + d)}{u_{n,m-1} + a_3}C_4 \tag{20}$$

Comments to List

While the equations 1-5 of the above list are pure analogues of the sin-Gordon equation, the equation 6 is Burgers type equation. The symmetries (19,20) depend on arbitrary constants C_1, C_2, C_3, C_4, which may equal zero, and this is natural for Burgers type equations. As well as in continuous case, the equation 6 is obtained from a linear equation by discrete analogue of the Hopf-Cole transform

$$u_{n,m} = \frac{v_{n+1,m}}{v_{n,m}}. \tag{21}$$

Non-autonomous linear equation

$$v_{n+1,m+1} = \alpha_{n,m} v_{n+1,m} + \beta_{n,m} v_{n,m+1} + \gamma_{n,m} v_{n,m}, \quad \alpha_{n,m}, \beta_{n,m}, \gamma_{n,m} \neq 0, \forall n, m \tag{22}$$

is transformed by (21) to

$$\begin{aligned} &(u_{n+1,m+1} - \beta_{n+1,m})(\alpha_{n,m} u_{n,m} + \gamma_{n,m}) u_{n,m+1} \\ &= (u_{n,m+1} - \beta_{n,m})(\alpha_{n+1,m} u_{n+1,m} + \gamma_{n+1,m}) u_{n,m}. \end{aligned} \tag{23}$$

The equation (6) is particular case of (23) corresponding to

$$\beta_{n,m} = -d, \quad \alpha_{n,m} = (-b_1)^n, \quad \gamma_{n,m} = a_3(-b_1)^n. \tag{24}$$

The generalized symmetries (19,20) are obtained from the following symmetries

$$\frac{d}{dt_1} v_{n,m} = C_1(-b_1)^{-m} v_{n+1,m} + C_2(-b_1)^m v_{n-1,m},$$
$$\frac{d}{dt_2} v_{n,m} = -C_3 v_{n,m+1} - C_4 v_{n,m-1}$$

of (22,24) by same transformation (21).

Equation with a non-standard symmetry structure.

The equation

$$u_{n+1,m+1}(u_{n,m} - u_{n,m+1}) - u_{n+1,m}(u_{n,m} + u_{n,m+1}) + 1 = 0 \quad (25)$$

satisfies all integrability conditions (14). It possesses one of symmetries of the form (6):

$$\frac{d}{dt_2} u_{n,m} = (-1)^n \frac{u_{n,m}^2 + u_{n,m-1}u_{n,m+1}}{u_{n,m-1} + u_{n,m+1}}. \quad (26)$$

However, there is no second symmetry of the form (6) and therefore no five point symmetry (3,4). For eq. (25) we have found a more complicated generalized symmetry:

$$\frac{d}{dt_1} u_{n,m} = h_{n,m}h_{n-1,m}(a_n u_{n+2,m} - a_{n-1}u_{n-2,m}), \quad (27)$$

$$h_{n,m} = 1 - 2u_{n+1,m}u_{n,m}, \quad a_{n+2} = a_n.$$

Equation (27) is a new example of Itoh-Narita-Bogoyavlensky type lattice. The symmetry depend on arbitrary two periodic function a_n, which can be presented in the form: $a_n = \tilde{a} + \hat{a}(-1)^n$.

The equation (27) have the following two conservation laws

$$\frac{d}{dt_1} p_{n,m}^{(i)} = (T_1 - 1) q_{n,m}^{(i)}$$

with densities:

$$p_{n,m}^{(1)} = \log h_{n,m}, \quad p_{n,m}^{(2)} = a_n h_{n+1,m} h_{n-1,m} - a_{n-1} h_{n,m} - 2 a_n u_{n+2,m} u_{n-1,m}.$$

Functions $q_{n,m}^{(i)}$ are found automatically. There is also the following generalized symmetry

$$\begin{aligned} \frac{d}{dt_3} u_{n,m} = h_{n,m} h_{n-1,m} (& b_n u_{n+4,m} h_{n+2,m} h_{n+1,m} - b_{n-1} h_{n-2,m} h_{n-3,m} u_{n-4,m} \\ & + 2 u_{n,m} (b_{n-1} u_{n+3,m} u_{n-2,m} h_{n+1,m} - b_n h_{n-2,m} u_{n+2,m} u_{n-3,m}) \\ & + 2 (2 u_{n+1,m} u_{n,m} u_{n-1,m} - u_{n+1,m} - u_{n-1,m}) (b_n u_{n+2,m}^2 - b_{n-1} u_{n-2,m}^2) \\ & + 2 u_{n,m} (b_{n-1} u_{n+1,m} u_{n-2,m} - b_n u_{n+2,m} u_{n-1,m})), \quad b_{n+2} = b_n. \end{aligned} \tag{28}$$

It can be checked that (28) is also a generalized symmetry of discrete equations (25).

Darboux integrable equations.

List

3 (Darboux integrable equations with N point integrals, such that $N \geq 3$)

1.

$$(u_{n+1,m+1}-u_{n+1,m})(u_{n,m}-u_{n,m+1})+u_{n+1,m+1}+u_{n+1,m}+u_{n,m+1}+u_{n,m}=0,$$

$$W_1=\frac{2(u_{n+1,m}+u_{n,m})+1}{(u_{n+2,m}-u_{n,m})(u_{n+1,m}-u_{n-1,m})},$$

$$W_2=(-1)^n\frac{u_{n,m+1}+u_{n,m-1}-2(u_{n,m}+1)}{u_{n,m+1}-u_{n,m-1}}$$

2. $b_2, |b_2^2-1|+|c_4|\neq 0,$

$$u_{n+1,m+1}(u_{n,m}+b_2u_{n,m+1})+u_{n+1,m}(b_2u_{n,m}+u_{n,m+1})+c_4=0,$$

$$W_1=\frac{u_{n+1,m}u_{n,m}(b_2^2-1)+b_2c_4}{(u_{n+2,m}-u_{n,m})(u_{n+1,m}-u_{n-1,m})},$$

$$W_2=(-1)^n\frac{b_2(u_{n,m+1}+u_{n,m-1})+2u_{n,m}}{u_{n,m+1}-u_{n,m-1}}$$

3.

$$(u_{n+1,m}+a_3 u_{n+1,m+1})(u_{n,m}+a_3 u_{n,m+1})+u_{n+1,m}+u_{n,m}+\frac{1}{a_3+1}=0,$$

$$W_1=(-a_3)^{-m}\frac{(a_3+1)(u_{n+1,m}+u_{n,m})+1}{(u_{n+2,m}-u_{n,m})(u_{n+1,m}-u_{n-1,m})},$$

$$W_2=\left(\frac{u_{n,m-1}+a_3 u_{n,m}+1}{\sqrt{-a_3}(u_{n,m}+a_3 u_{n,m+1})}\right)^{(-1)^n}$$

4. $b_3=-1$ Discrete Liuville equation Hirota 79

$$u_{n+1,m+1}u_{n,m}+b_3 u_{n+1,m}u_{m,n+1}+1=0, \quad b_3=\pm 1,$$

$$W_1=(-b_3)^m\frac{u_{n+1,m}-b_3 u_{n-1,m}}{u_{n,m}},$$

$$W_2=(-b_3)^n\frac{u_{n,m+1}-b_3 u_{n,m-1}}{u_{n,m}}$$

5. Discrete Liuville equation Hirota 87

$$(u_{n+1,m+1}+1)(u_{n,m}+1)-(u_{n+1,m}-1)(u_{n,m+1}-1)=0,$$

$$W_1=\frac{u_{n,m}^2-1}{(u_{n+1,m}+u_{n,m})(u_{n-1,m}+u_{n,m})},$$

$$W_2=\frac{u_{n,m}^2-1}{(u_{n,m+1}+u_{n,m})(u_{n,m-1}+u_{n,m})}$$

6.

$$(u_{n+1,m+1}-1)(u_{n,m}+1)u_{n,m+1}+(u_{n,m+1}-1)(u_{n+1,m}+1)u_{n,m}=0,$$

$$W_1=(-1)^m\frac{(u_{n+1,m}u_{n,m}-1)u_{n-1,m}}{u_{n-1,m}u_{n,m}-1},$$

$$W_2=(-1)^n\frac{(u_{n,m}+u_{n,m+1})(u_{n,m-1}+1)}{(u_{n,m+1}-1)(u_{n,m-1}+u_{n,m})}$$

List

4 (The case when one of first integrals is two point one)

1.

$$u_{n+1,m+1}u_{n,m+1} + b_2(u_{n+1,m} + d)(u_{n,m} + d) = 0, \quad b_2 \neq 0,$$

$$W_1 = (-b_2)^{-m}(u_{n+2,m} - u_{n,m})(u_{n+1,m} - u_{n-1,m}),$$

$$W_2 = \left(\frac{\sqrt{-b_2}(u_{n,m} + d)}{u_{n,m+1}}\right)^{(-1)^n}$$

2. Adler Startsev 99

$$u_{n,m}u_{n,m+1}(u_{n+1,m+1}+b_1u_{n+1,m})+u_{n,m}+b_1u_{n,m+1} = 0, \quad b_1^2 = 1,$$

$$W_1 = (-b_1)^m \frac{u_{n+1,m}u_{n,m} + 1}{u_{n,m}},$$

$$W_2 = \frac{(u_{n,m+2} + b_1u_{n,m+1})(u_{n,m} + b_1u_{n,m-1})}{(u_{n,m+2} - u_{n,m})(u_{n,m+1} - u_{n,m-1})}$$

3. Möbius transformation of example from Startsev 2010

$$u_{n+1,m+1}u_{n,m+1} - u_{n+1,m}u_{n,m} + u_{n,m+1} - u_{n,m} = 0,$$

$$W_1 = u_{n,m}(u_{n+1,m} + 1),$$

$$W_2 = \frac{(u_{n,m+2} - u_{n,m})(u_{n,m+1} - u_{n,m})}{(u_{n,m+2} - u_{n,m+1})(u_{n,m} - u_{n,m-1})}$$

4.

$$u_{n+1,m+1}u_{n,m} - u_{n+1,m}u_{n,m+1} + u_{n+1,m} - u_{n,m} = 0,$$

$$W_1 = \frac{u_{n+1,m} - u_{n,m}}{u_{n,m} - u_{n-1,m}}, \qquad W_2 = \frac{u_{n,m+1} - 1}{u_{n,m}}$$

5.

$$(u_{n+1,m+1} - u_{n+1,m} + b_4)(u_{n,m+1} - u_{n,m} + b_4) = d^2, \quad d \neq 0,$$

$$W_1 = u_{n+1,m} - u_{n-1,m},$$

$$W_2 = (-1)^n \frac{u_{n,m+1} - u_{n,m} + b_4 - d}{u_{n,m+1} - u_{n,m} + b_4 + d}$$

6.

$$u_{n+1,m+1}(a_2 u_{n,m} + u_{n,m+1}) + a_2^2 u_{n+1,m} u_{n,m} = 0, \quad a_2 \neq 0,$$

$$W_1 = a_2^{-3m} u_{n+1,m} u_{n,m} u_{n-1,m},$$

$$W_2 = (d)^{-n} \frac{a_2 u_{n,m} - d u_{n,m+1}}{u_{n,m+1} - a_2 d u_{n,m}}, \quad d = -\frac{1+\sqrt{3}i}{2}, \quad i^2 = -1$$

7.

$$u_{n+1,m+1}(a_2 u_{n,m} + u_{n,m+1}) + u_{n+1,m}(a_2^2 u_{n,m} - a_2 u_{n,m+1}) = 0, \quad a_2 \neq 0,$$

$$W_1 = (-a_2^2)^{-m} u_{n+1,m} u_{n-1,m},$$

$$W_2 = i^n \frac{a_2 u_{n,m} + i u_{n,m+1}}{u_{n,m+1} + a_2 i u_{n,m}}, \quad i^2 = -1$$

8.

$$u_{n+1,m+1}(a_2 u_{n,m} + u_{n,m+1}) + u_{n+1,m}((a_2^2 - d^2) u_{n,m} + a_2 u_{n,m+1}) = 0, \quad d \neq 0,$$

$$W_1 = \frac{u_{n+1,m}}{u_{n-1,m}}, \qquad W_2 = (-1)^n \frac{(a_2 + d) u_{n,m} + u_{n,m+1}}{u_{n,m+1} + (a_2 - d) u_{n,m}}$$

9.

$$u_{n+1,m+1}u_{n,m+1} + u_{n+1,m}u_{n,m} + c_4 = 0,$$

$$W_1 = (-1)^m(2u_{n+1,m}u_{n,m} + c_4), \qquad W_2 = \left(\frac{u_{n,m+1}}{u_{n,m-1}}\right)^{(-1)^n}$$

Comment The first integrals W of the form $W = \phi^{(-1)^n}$ can be rewritten in the form

$$\hat{W} = (-1)^n\hat{\phi}, \quad \hat{\phi} = \frac{1+\phi}{1-\phi},$$

see examples eq.3 of List 3, eqs. 1,9 of List 4.

Equations linearizable via a 2-point first integral

A 2-point first integrals in the second direction (or in the m-direction) which are of the form:

$$(T_1 - 1)W_2 = 0, \quad W_2 = w_n^{(2)}(u_{n,m}, u_{n,m+1}). \tag{29}$$

If equation (2) *equivalent* to relations (29), then it equivalent to $W_2 = \kappa_m$. For example, if equation (2) can be written in the form

$$T_1\phi = \alpha\phi + \beta, \quad \phi = \frac{\nu_1 u_{n,m+1} + \nu_2 u_{n,m} + \nu_3}{\nu_4 u_{n,m+1} + \nu_5 u_{n,m} + \nu_6}, \quad \alpha \neq 0 \tag{30}$$

then it has the first integral in the m-direction:

$$W_2 = \begin{cases} \phi + n\beta, & \alpha = 1 \\ \alpha^{-n}\left(\phi + \frac{\beta}{\alpha-1}\right), & \alpha \neq 1 \end{cases}$$

Equation $W_2 = \kappa_m$ can be rewritten in the form:

$$u_{n,m+1} + \mu_{n,m}u_{n,m} + \eta_{n,m} = 0. \tag{31}$$

So equation (30) *equivalent* non-autonomous non-homogeneous linear equation (31).

For any equation of List 5 below, there is a relation of the form

$$T_1\phi = \frac{\delta_1\phi + \delta_2}{\delta_3\phi + \delta_4}, \qquad (32)$$

shown in the list, and the function ϕ has the form shown in eq. (30). By using an autonomous linear-fractional transformation of the function ϕ, we reduce the relation (32) to the form (30), and that transformation changes in the formula for ϕ the coefficients ν_j only. So, all equations of List 5 are equivalent to a linear equation (31).

Equations of the following list are defined by some relationships for the coefficients a_j, b_j, c_j of eq. (2), and all the equations have the following restriction:

$$a_1 = b_1 = c_1 = 0.$$

List

5 (equations possessing a 2-point first integral of the form (29))

1. $a_3 \neq 0, \quad a_2 \neq b_3, \quad a_4 \neq c_3, \quad a_2 c_3 \neq a_3 c_2,$

$$b_2 = \frac{a_3 c_2 (a_2 - b_3) + a_2 (b_3 a_4 - a_2 c_3)}{a_3 (a_4 - c_3)},$$

$$b_4 = \frac{a_3 c_2 - c_3 a_2 + a_4 b_3}{a_3},$$

$$c_4 = \frac{a_3 c_2 (a_4 - c_3) + c_3 (a_2 c_3 - b_3 a_4)}{a_3 (a_2 - b_3)}.$$

$$T_1 \phi = \frac{(a_2 - b_3)(a_2 c_3 - a_3 c_2)}{(\phi + a_2)(a_4 - c_3)} - b_3, \quad \phi = \frac{u_{n,m+1} a_3 (a_2 - b_3) + c_3 a_2 - a_4 b_3}{u_{n,m} (a_2 - b_3) + a_4 - c_3}$$

2. $a_3 \neq 0, \quad b_3 \neq 0, \quad a_3 c_4 \neq a_4 c_3,$

$$a_2 = b_3, \quad b_2 = \frac{b_3^2}{a_3} \quad b_4 = \frac{a_4 b_3}{a_3}, \quad c_2 = \frac{c_3 b_3}{a_3}.$$

$$T_1 \phi = -\frac{c_3 \phi + c_4 a_3}{\phi + a_4.}, \quad \phi = u_{n,m} b_3 + u_{n,m+1} a_3.$$

3. $a_2 \neq 0, \quad b_3 \neq 0, a_3 = 0, \quad c_3 = \frac{a_4 b_3}{a_2},$

$$b_4 = \frac{a_2(b_3 c_2 + a_4 b_2) - a_4 b_2 b_3}{a_2^2}, \quad c_4 = \frac{a_4(a_2 c_2(b_3 + a_2) - a_4 b_2 b_3)}{a_2^3}$$

$$T_1\phi = -\phi \frac{b_3}{a_2} - b_2, \quad \phi = \frac{u_{n,m+1} a_2^2 + c_2 a_2 - a_4 b_2}{u_{n,m} a_2 + a_4}.$$

4. $a_2^2 \neq a_3 b_2,$

$$a_4 = c_3, \quad b_3 = a_2 \quad b_4 = c_2, \quad c_4 = \frac{2 a_2 c_2 c_3 - c_3^2 b_2 - a_3 c_2^2}{a_2^2 - b_2 a_3}.$$

$$T_1\phi = -\frac{a_2\phi + b_2}{a_3\phi + a_2}, \quad \phi = \frac{u_{n,m+1}(a_2^2 - a_3 b_2) + c_2 a_2 - b_2 c_3}{u_{n,m}(a_2^2 - a_3 b_2) + c_3 a_2 - a_3 c_2}.$$

5. $a_3 \neq 0, \quad a_2 b_3 \neq a_3 b_2,$

$$a_4 = c_3, \quad b_4 = \frac{c_3 b_3}{a_3} \quad c_2 = \frac{c_3 a_2}{a_3}, \quad c_4 = \frac{c_3^2}{a_3}.$$

$$T_1\phi = -\frac{b_3\phi + b_2 a_3}{\phi + a_2}, \quad \phi = \frac{u_{n,m+1} a_3 + c_3}{u_{n,m}}.$$

For example, any equation of the form

$$u_{n+1,m+1}(a_2 u_{n,m} + a_3 u_{n,m+1}) + u_{n+1,m}(b_2 u_{n,m} + b_3 u_{n,m+1}) = 0, \quad a_2 b_3 \neq a_3$$

is a particular case of eq. 3 (if $a_3 = 0$) or of eq. 5 (if $a_3 \neq 0$) of List 5. Eqs. 6,7 and 8 of List 4 are of this form too.

Theorem

A nondegenerate and nonlinear equation (2) *has a first integral* (29) *if and only if it belongs to List 5. Any equation of List 5 is equivalent to a linear equation of the form* (31)*.*

Here a first integral is of the form:

$$(T_2 - 1)W_1 = 0, \quad W_1 = w_m^{(1)}(u_{n,m}, u_{n+1,m}). \tag{33}$$

An equation (2), possessing such first integral, is *equivalent* to the relation (33) and hence to the equation $W_1 = \kappa_n$.
For example, if there are the relations

$$T_2\phi = \alpha\phi + \beta, \quad \phi = \frac{\nu_1 u_{n+1,m}u_{n,m} + \nu_2 u_{n+1,m} + \nu_3 u_{n,m} + \nu_4}{\nu_5 u_{n,m} + \nu_6}, \quad \alpha \neq 0, \tag{34}$$

with constant coefficients α, β, ν_j, then W_1 has the form:

$$W_1 = \begin{cases} \phi + m\beta, & \alpha = 1; \\ \alpha^{-m}\left(\phi + \frac{\beta}{\alpha - 1}\right), & \alpha \neq 1. \end{cases} \tag{35}$$

Moreover, the equation $W_1 = \kappa_n$ can be rewritten in the form:

$$\nu_1 u_{n+1,m}u_{n,m} + \nu_2 u_{n+1,m} + \hat{\mu}_{n,m}u_{n,m} + \hat{\eta}_{n,m} = 0.$$

After non-autonomous Möbius transformation of $u_{n,m}$ the last equation can be expressed as:

$$\hat{u}_{n+1,m} + \mu_{n,m}\hat{u}_{n,m} + \eta_{n,m} = 0. \tag{36}$$

List

6 (equations possessing a 2-point first integral of the form (33))

1. $a_3 \neq 0, \quad b_1 \neq 0, \quad b_2 \neq 0, \quad |a_3 - a_2| + |b_1 c_4 - c_2 b_2| \neq 0,$

$$a_1 = \frac{b_1 a_3}{b_2}, \quad a_4 = \frac{a_2 b_2}{b_1}, \quad b_3 = \frac{a_2 b_2}{a_3}, \quad b_4 = \frac{a_2 b_2^2}{b_1 a_3},$$

$$c_1 = \frac{b_1(b_2^2 c_2 + a_3 b_2 c_2 - a_3 b_1 c_4)}{b_2^3}, \quad c_3 = \frac{b_1 b_2 c_4 + a_3 b_2 c_2 - a_3 b_1 c_4}{b_2^2}.$$

$$T_2 \phi = -\phi \frac{b_2}{a_3} - b_2 c_2 - a_3 c_2 + \frac{b_1 a_3 c_4}{b_2},$$

$$\phi = \frac{u_{n+1,m} u_{n,m} b_1 b_2 a_3 + u_{n+1,m} a_2 b_2^2 + a_3(b_1 c_4 - b_2 c_2)}{u_{n,m} b_1 + b_2}.$$

2. $a_3 \neq 0, \quad b_2 \neq 0, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0,$

$$a_4 = \frac{b_4 a_3}{b_2}, \quad b_3 = 0, \quad c_1 = 0, \quad c_3 = \frac{c_2 a_3}{b_2}.$$

$$T_2 \phi = -\frac{b_2}{a_3}(\phi + c_4), \quad \phi = u_{n+1,m} u_{n,m} b_2 + u_{n+1,m} b_4 + u_{n,m} c_2.$$

3. $a_4 \neq 0, \quad b_3 \neq 0, \quad b_4 \neq 0,$

$$a_1 = 0, \quad a_2 = \frac{a_4 b_3}{b_4}, \quad a_3 = 0, \quad b_1 = 0, \quad b_2 = 0,$$

$$c_2 = \frac{c_1 b_4^2 + b_3 a_4 c_3 - a_4 b_4 c_1}{b_3 b_4}, \quad c_4 = \frac{b_3 b_4 c_3 + a_4 b_3 c_3 - a_4 b_4 c_1}{b_3^2}$$

$$T_2 \phi = -\frac{b_4}{a_4}(\phi + c_1), \quad \phi = \frac{u_{n+1,m} b_3^2 + b_3 c_3 - c_1 b_4}{u_{n,m} b_3 + b4}.$$

4. $a_1 \neq 0, \quad b_1 \neq 0, \quad |b_3| + |c_3| \neq 0, \quad a_3 = 0, \quad a_4 = 0,$

$$a_2 = \frac{a_1 b_3}{b_1}, \quad b_2 = 0, \quad b_4 = 0, \quad c_2 = \frac{a_1 c_3}{b_1}, \quad c_4 = 0.$$

$$T_2 \phi = -\frac{b_1}{a_1}(\phi + c_1), \quad \phi = \frac{u_{n+1,m} u_{n,m} b_1 + u_{n+1,m} b_3 + c_3}{u_{n,m}}$$

5. $a_2 \neq 0, \quad b_3 \neq 0, \quad a_1 = 0, \quad a_3 = 0, \quad a_4 = 0,$

$$b_1 = 0, \quad b_2 = 0, \quad b_4 = 0, \quad c_2 = \frac{a_2 c_3}{b_3}, \quad c_4 = 0.$$

$$T_2 \phi = -\frac{b_3}{a_2}(\phi + c_1), \quad \phi = \frac{u_{n+1,m} b_3 + c_3}{u_{n,m}}.$$

For example, eqs. 2,3 and 9 of List 4 are the particular cases of equations presented in List 6.

Theorem

A nondegenerate and nonlinear equation of the form (2) *has a first integral defined by* (33) *if and only if it belongs to List 6. Any equation of List 6 is equivalent to a linear equation of the form* (36) *up to a non-autonomous Möbius transformation of its solutions* $u_{n,m}$.

Additional results

Viallet equations

This equation is defined by following conditions for (1):

$$F(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = F(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}) \\ = F(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n+1,m}),$$

and depends on 7 arbitrary constant parameters. As it has been shown that the Q_V equation has generalized symmetries (6) for all values of these parameters. Intersection of our class (2) with Q_V has next form:

$$(u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1})k_1 + (u_{n,m}u_{n+1,m+1} + u_{n+1,m}u_{n,m+1})k_2 \\ + (u_{n,m} + u_{n+1,m} + u_{n,m+1} + u_{n+1,m+1})k_3 + k_4 = 0. \quad (37)$$

If $k_1 = k_2 = 0$, then this equation is linear.

Theorem

The equation (37) with $k_1 \neq 0$ or $k_2 \neq 0$ is Darboux integrable or degenerate or equivalent to linear.

Proof. Considering all possible cases it is easy to check that eq. (37) is degenerate or is equivalent up to linear transformation $\hat{u}_{n,m} = \alpha u_{n,m} + \beta$ to (11) or to one of equations of 1,2,4 of List 3 and 9 of List 4.∎

Xenitidis

The Q_V equation is generalized by a class of polylinear equations given in Xenitidis 09 by

$$\begin{aligned} F(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) &= \pi_1 F(u_{n+1,m}, u_{n,m}, u_{n+1,m+1}, u_{n,m+1}) \\ &= \pi_2 F(u_{n,m+1}, u_{n+1,m+1}, u_{n,m}, u_{n+1,m}), \quad \pi_1 = \pm 1,\ \pi_2 = \pm 1. \end{aligned}$$

All equations also have the generalized symmetries (6). In addition to the equation Q_V, we have here two other possible cases (up to transformation $n \leftrightarrow m$):

$$\begin{aligned} &(u_{n+1,m}u_{n,m+1}u_{n+1,m+1} + u_{n,m}u_{n,m+1}u_{n+1,m+1} - u_{n,m}u_{n+1,m}u_{n+1,m+1} \\ &\qquad - u_{n,m}u_{n+1,m}u_{n,m+1})k_1 + (u_{n,m}u_{n+1,m} - u_{n,m+1}u_{n+1,m+1})k_2 \\ &\qquad + (u_{n,m} + u_{n+1,m} - u_{n,m+1} - u_{n+1,m+1})k_3 = 0, \end{aligned}$$

$$\begin{aligned} &(u_{n+1,m}u_{n,m+1}u_{n+1,m+1} - u_{n,m}u_{n,m+1}u_{n+1,m+1} - u_{n,m}u_{n+1,m}u_{n+1,m+1} \\ &\qquad + u_{n,m}u_{n+1,m}u_{n,m+1})k_1 + (u_{n,m}u_{n+1,m+1} - u_{n+1,m}u_{n,m+1})k_2 \\ &\qquad + (u_{n,m} - u_{n+1,m} - u_{n,m+1} + u_{n+1,m+1})k_3 = 0. \end{aligned}$$

Theorem

Any Xenitidis equation of both forms is equivalent to linear equation up to an autonomous point transformation.

Proof. Both classes are invariant under autonomous Möbious transformations. Using Möbious transformation we always can make $k_1 = 0$ and $k_2 k_3 = 0$ in both equations. So we get a linear equation or one of equations (11).∎