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Mathematik

Uniform Polytime Computable Operators and Data Structures for Real Analytic Functions

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Theory and Practice

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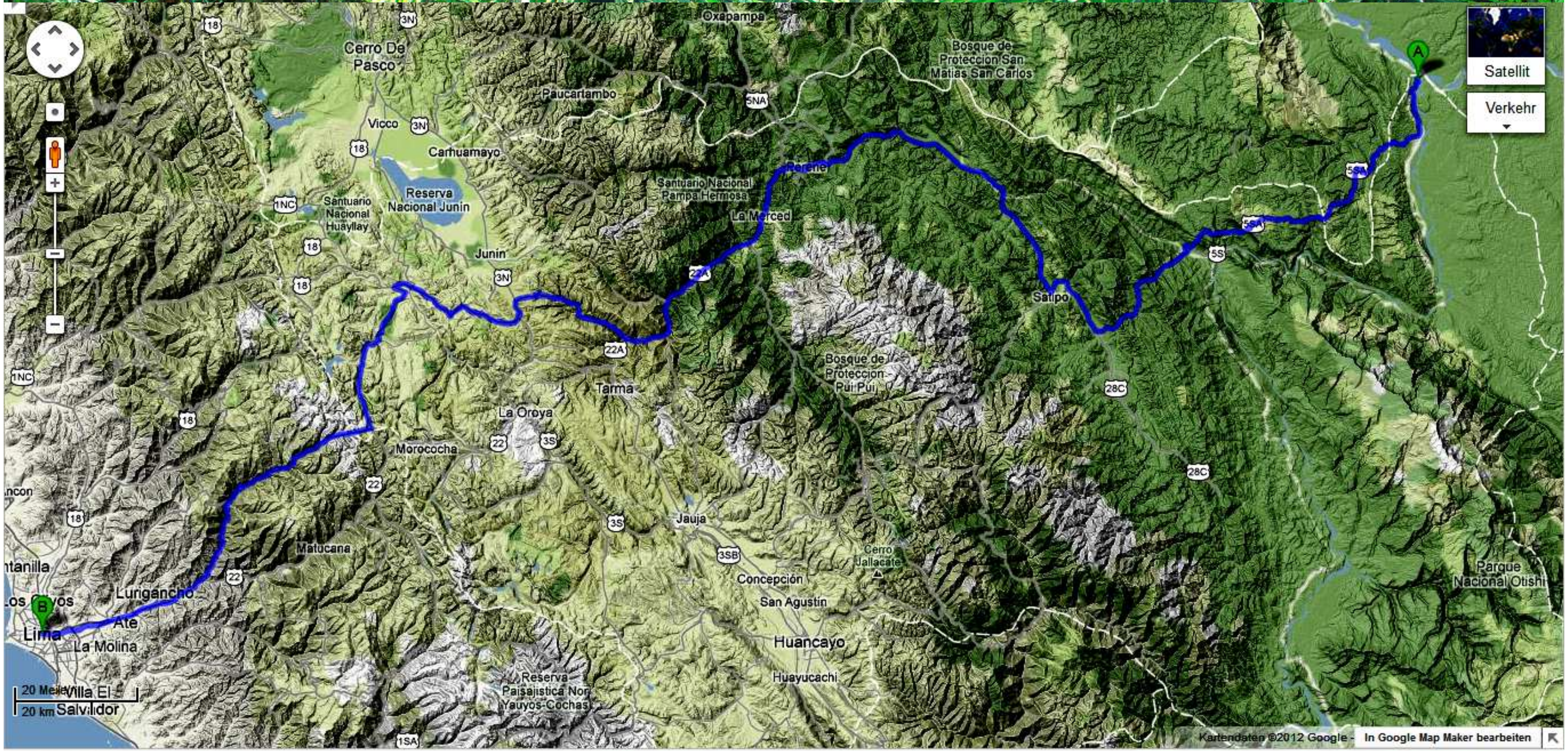
Maximizing Analytic
Functions in Polytime



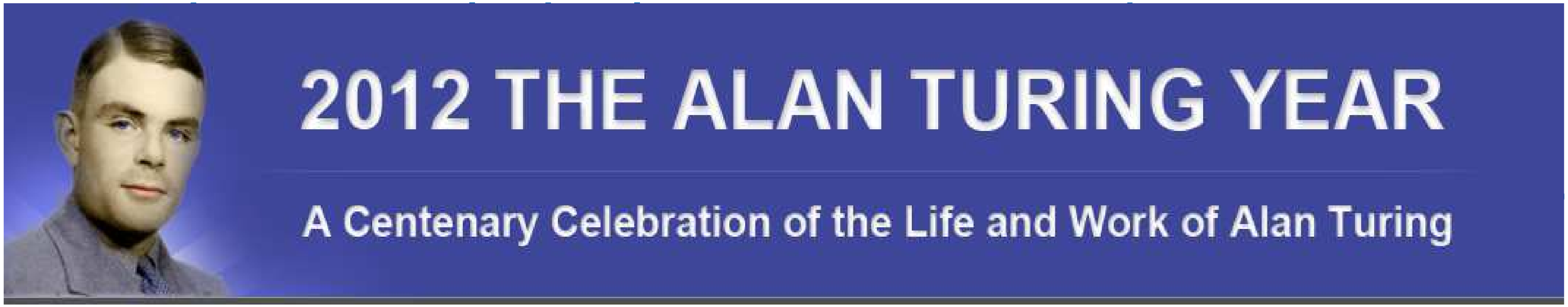
Unbounded Precision Computation

incl. asym. runtime

advertisement

Functions in Polytime

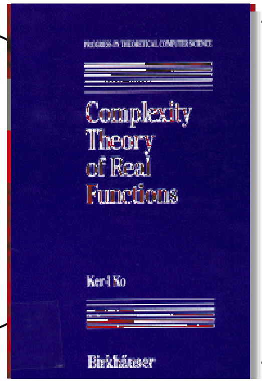
- guaranteed behavior, closed under composition



Computability and Complexity in Analysis (CCA)

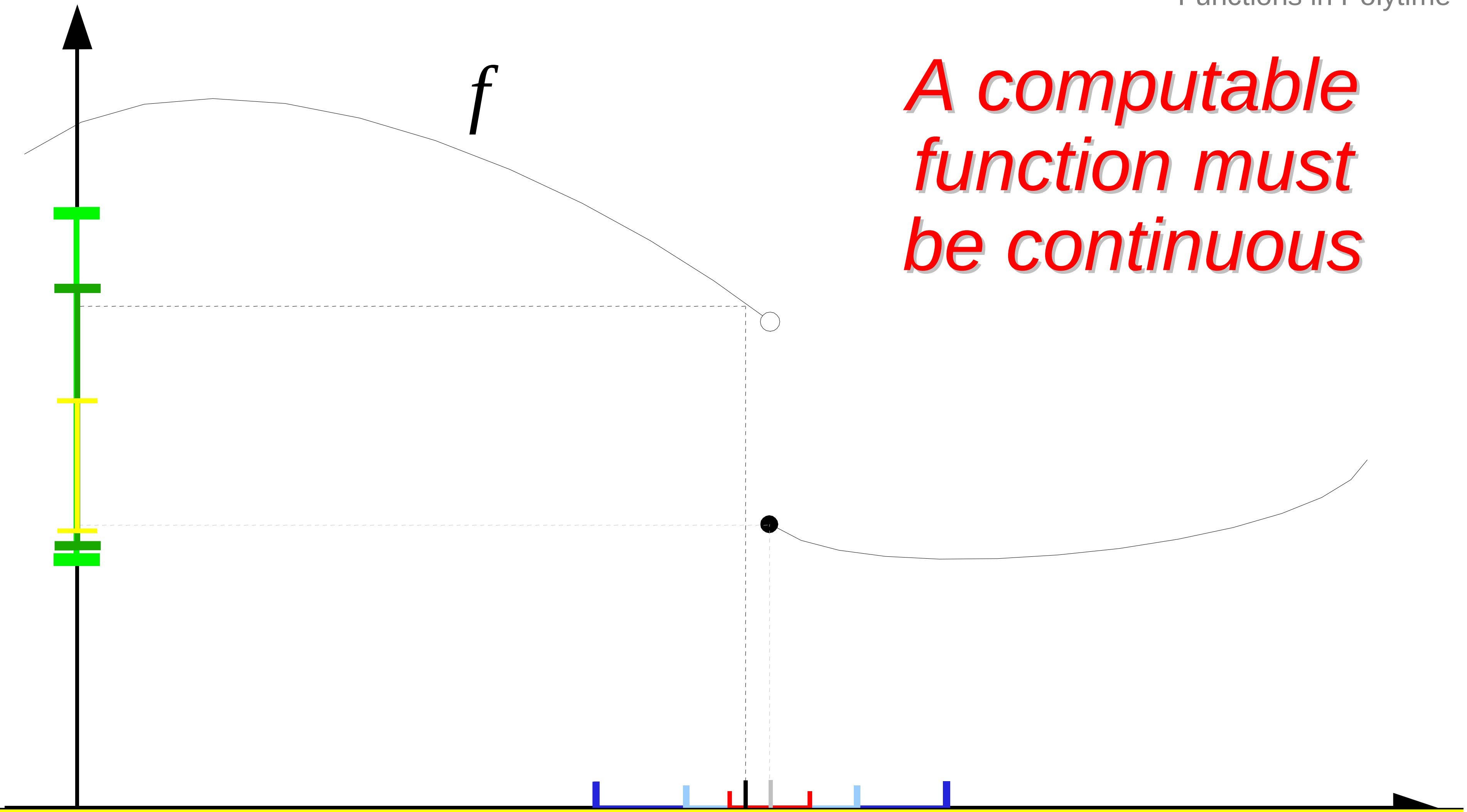
Theory of (approximate) real computing with prescribable absolute output error 2^{-n}.

Recursion Theory, (qual.) Topology "*Weihrauch School*"



e.g. "$\mathcal{P}$ *versus* $\mathcal{NP}$", quant. Topology Stephen A. Cook's last two PhD students

Computable Real Functions



A computable function must be continuous

$x \in \mathbb{R}$ computable $\Leftrightarrow |x - a_n/2^{n+1}| \leq 2^{-n}$ for recursive $(a_n) \subseteq \mathbb{Z}$

Real Function Complexity

iRRAM (GMP/MPFR)

Function $f:[0,1]\to\mathbb{R}$ computable in time $t(n)$
if some TM can, on input of $n\in\mathbb{N}$ and of
$(a_m)\subseteq\mathbb{Z}$ with $|x-a_m/2^{m+1}|<2^{-m}$ ($\equiv_p$ ρ_{sd}-name)
in time $t(n)$ output $b\in\mathbb{Z}$ with $|f(x)-b/2^{n+1}|<2^{-n}$.

Examples: a) $+, \times, \exp$ polytime on $[0,1]$!

b) $f(x)\equiv\sum_{n\in L} 4^{-n}$ iff $L\subseteq\{0,1\}^*$ polytime-decidable

c) $1/\ln(e)$ polytime computable / sgn, Heaviside not computable XSC ?

Observation i) If f computable $\Rightarrow$ continuous.
ii) If f computable in time $t(n)$, then
$t(n+2)$ is a modulus of uniform continuity of f.

$\mathbb{D}_n := \{\, k/2^n : k\in\mathbb{Z} \,\}$, $\mathbb{D} = \bigcup_n \mathbb{D}_n$ dyadic rationals

Effects in Real Complexity

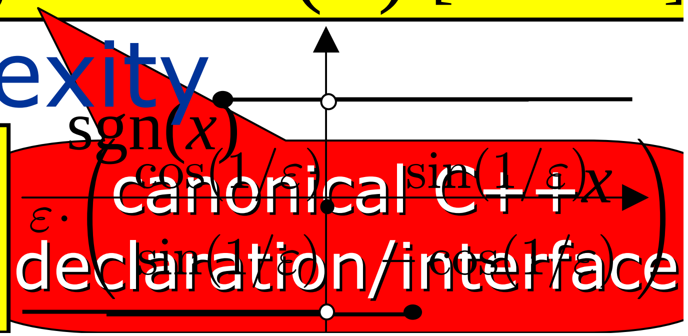
- Consider multivalued 'functions' with
- additional <u>discrete</u> data ('enrichment').

Example c1): $\exp$ not computable on entire $\mathbb{R}$,
c2) Evaluation $(f,x)\to f(x)$ is not computable in time depending <u>only</u> on output precision n.

Example b): Given real symmetric $d\times d$ matrix A, find <u>an</u> eigenvector: incomputable; but computable when knowing Card $\sigma(A)$ [Z+B'04]

- parameterized real complexity

Example a): Tests for in-/equality are undecidable



canonical C++ declaration/interface

Nonuniform Complexity of Operators

f:[0,1]→[0,1] polytime computable (⇒ continuous)

- Max: $f \to \mathrm{Max}(f): x \to \max\{\, f(t): t \le x\}$

  Max(f) computable in exponential time;
  polytime-computable iff $\mathcal{P}=\mathcal{NP}$

  Shary/ Kreinovich

- $\int: f \to \int f: x \to \int_0^x f(t)\,dt$

  non-uniform

  $\int f$ computable in exponential time;
  polytime-computable iff $\mathcal{P}=\#\mathcal{P}$

  even when restricting to $f \in C^\infty$
  but for analytic f polytime

  [Friedman&Ko'80ies]

- dsolve: $C[0,1]\times[-1,1] \ni f \to z: \quad \dot z(t)=f(t,z),\; z(0)=0.$

  in general no computable z

  for $f \in C^1$ polytime-computable iff $\mathcal{P}=\mathcal{PSPACE}$

  for $f \in C^k$ between $\mathcal{CH}$ and $\mathcal{PSPACE}$

Representing Power Series

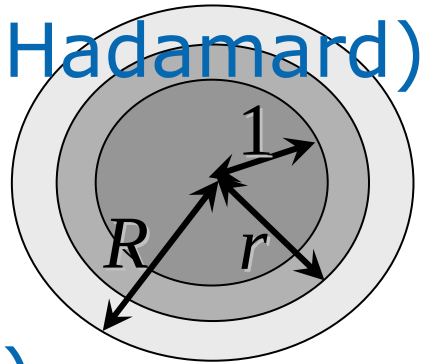
$\sum_j c_j z^j$

incomputable [ZhWe'01]

- radius of convergence $R = 1/\limsup_j |c_j|^{1/j}$
- to $0<r<R$ exist $C\in\mathbb{N}$: $|c_j| \leq C/r^j$ (Cauchy-Hadamard)
- $\mathbb{N} \ni K :\geq 1/\log(r) = \Theta(1/(r-1))$
- tail bound $|\sum_{j\geq N} c_j z^j| \leq C\cdot(|z|/r)^N/(1-|z|/r)$

binary

unary



Complexity uniform in $|z|\leq 1$: (i.e. $R>1$)

Convergence degrades es as $r\to 1$; quantitatively?

Theorem 1: Represent series $\sum_j c_j z^j$ with $R>1$ as [a $(\rho_{sd})^\omega$-name of] (c_j) and $K, C\in\mathbb{N}$ as above.

The following are uniformly computable in time polyn. in $n+K+\log(C)$: i) eval, ii) sum, iii) product, iv) derivative, v) anti-derivative, vi) max on $[a,b]$

Uniformly Polynomial Time Operations on Power Series

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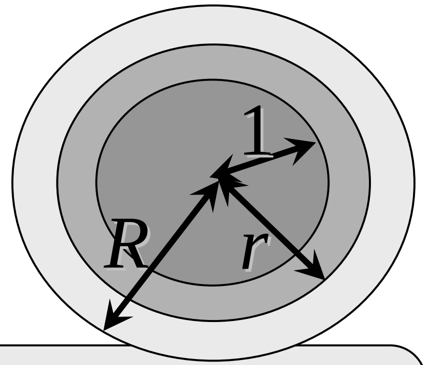
$\sum_j c_j z^j$

- radius of convergence $R = 1/\limsup_j |c_j|^{1/j}$
- to $0<r<R$ exist $C\in\mathbb{N}$: $|c_j| \le C/r^j$
- $\mathbb{N} \ni K :\geq 1/\log(r) = \Theta(1/(r-1))$
- tail bound $|\sum_{j\ge N} c_j z^j| \le C\cdot(|z|/r)^N/(1-|z|/r)$

binary

unary



Proof (Sketch, i): $r \ge 2^{1/K}$

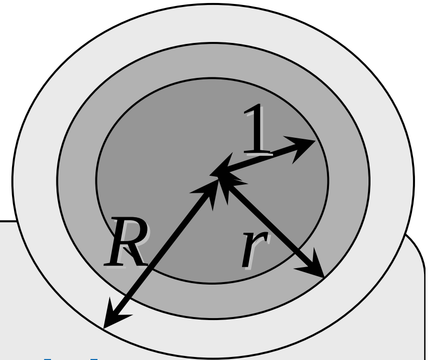
$N \approx K\cdot(n+\log K)+\log C$ $\quad |\sum_{j\ge N} c_j z^j| \le C\cdot 2^{-N/K}\cdot r\cdot K \le 2^{-n}$

Theorem 1: Represent series $\sum_j c_j z^j$ with $R>1$ as [a $(\rho_{sd})^\omega$-name of] (c_j) and $K,C\in\mathbb{N}$ as above. The following are uniformly computable in time polyn. in $n+K+\log(C)$: **i) eval** ii) sum, iii) product, iv) derivative, v) anti-derivative, vi) max on [*a*,*b*]

Uniformly Polynomial Time Operations on Power Series

$\sum_j c_j z^j$

- radius of convergence $R=1/\limsup_j |c_j|^{1/j}$
- to $0<r<R$ exist $C\in\mathbb{N}$: $|c_j|\leq C/r^j$ (binary)
- $\mathbb{N} \ni K :\geq 1/\log(r) = \Theta(1/(r-1))$ (unary)



Proof (Sketch, iv): $c_j'=(j+1)\cdot c_{j+1}$

$|c_j'|\leq C'/r^j$ *C'* not continuously computable

$|c_j'|\leq C'/\sqrt{r}^j$ $C':\geq C\cdot(1+2K/\mathrm{e}\cdot\ln 2)$ $K':=2K$

Theorem 1: Represent series $\sum_j c_j z^j$ with $R>1$ as [a $(\rho_{sd})^\omega$-name of] (c_j) and $K,C\in\mathbb{N}$ as above. The following are uniformly computable in time polyn. in $n+K+\log(C)$: i) eval, ii) sum, iii) product, **iv) derivative** v) anti-derivative, vi) max on [*a*,*b*]

Uniformly Polynomial Time Operations on Power Series

$\sum_j c_j z^j$

- radius of convergence $R=1/\limsup_j |c_j|^{1/j}$
- to $0<r<R$ exist $C\in\mathbb{N}$: $|c_j|\leq C/r^j$

rat. approx to b

Proof (Sketch, vi): $\sum_{j\leq N} r_j x^j$ approx to $2^{-n-1}=\varepsilon/2$

$\Phi(\underline{a},\underline{b},\underline{y},\underline{r}_0,\ldots,\underline{r}_N) :=$ "$\exists x:\ x\geq\underline{a}\ \wedge\ x\leq\underline{b}\ \wedge\ \sum_{j\leq N}\underline{r}_j\, x^j\geq\underline{y}$"

$\exists$-quant. FO sentence over $\mathbb{R}$ with $\mathbb{Z}$ coefficients

[Tarski] $\Leftrightarrow$ quant.-free FO Ψ comput. in poly.time

Theorem 1: Represent series $\sum_j c_j z^j$ with $R>1$ as [a $(\rho_{sd})^\omega$-name of] (c_j) and $K,C\in\mathbb{N}$ as above. The following are uniformly computable in time polyn. in $n+K+\log(C)$: i) eval, ii) sum, iii) product, iv) derivative, v) anti-derivative **vi) max on [*a*,*b*]**

Real Analytic Functions on [0,1]

Definition: $C^\omega[0,1] := \{ f:[0,1]\to\mathbb{R}$ restriction of complex differentiable $g:U\to\mathbb{C}$, $[0,1]\subseteq U\subseteq\mathbb{C}$ open $\}$

- real sequence $f(d)_{d\in\mathbb{D}\cap[0,1]}$
- $L\in\mathbb{N}$ binary: $R_L \subseteq U$
- $G\in\mathbb{N}$ unary $\forall z\in R_L$: $|g'(z)|\le G$.

2nd order representation

Theorem 2: These are mutually polyn.-time equivalent

Equival.: $f\in C^\infty[0,1]$ and $\exists F,L\in\mathbb{N}$ $\forall x\ \forall j$: $|f^{(j)}(x)|\le F\cdot L^j$

- real sequence $f(d)_{d\in\mathbb{D}}$ and $F\in\mathbb{N}$ binary, $L\in\mathbb{N}$ unary

Equival: f finitely many local power series on $[0,1]$

$\sum_j c_{j,m}\cdot(z-x_m)^j$, $m=1\ldots M$ unary binary $C_m,K_m\in\mathbb{N}$: $|c_{j,m}|\le C_m/r_m^{\,j}$

Theorem 3: On $C^\omega[0,1]$, i) eval ii) sum … vi) max are computable within parameterized polyn. time

Conclusion and Perspectives

For the space of real analytic functions, presented

- uniform strengthenings of previous algorithms
- with parameterized upper complexity bounds
- and specification of (additional discrete) data i.e. function interface declarations

- Actually implement the algorithms
- Quantitatively refine parameterized polytime upper complexity bounds
- multivariate power series? Gevrey?
- Devise new approaches (e.g. to solving ODEs)
- Collaboration of CCA and Interval communities!

Remember Dagstuhl 2006

Reliable Implementation of Real Number Algorithms: Theory and Practice