# A numerical verification method for solutions to systems of elliptic partial differential equations

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# 1. Introduction

We consider systems of elliptic partial differential equations:

$$\begin{cases} -\varepsilon^2 \Delta u = f(u) - \delta v, & \text{in } \Omega, \\ -\Delta v = u - \gamma v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

Here

$\Omega$ is bounded polygonal domain in $\mathbb{R}^2$ .

$\varepsilon \neq 0, \gamma$ and $\delta$ are real parameters.

A mapping $f : H_0^1(\Omega) \to L^2(\Omega)$

# 1. Introduction

We denote $L^2$-inner product and $H_0^1$ -inner product as

$$
\begin{aligned}
(u, w)_{L^2} &:= \int_\Omega uw dx, \\
(\nabla u, \nabla w)_{L^2} &:= \int_\Omega \nabla u \cdot \nabla w dx.
\end{aligned}
$$

# 1. Introduction

For the system (1), we have weak form:

Find $u, v \in H_0^1(\Omega)$, such that

$$(\nabla u, \nabla w)_{L^2} = \frac{1}{\varepsilon^2}\left((f(u), w)_{L^2} - \delta(v, w)_{L^2}\right), \tag{2}$$

$$(\nabla v, \nabla w)_{L^2} = (u, w)_{L^2} - \gamma(v, w)_{L^2}, \forall w \in H_0^1(\Omega). \tag{3}$$

# 1. Introduction

When $u$ is known function, equation (3) has unique solution.

Then, $v$ is presented by $v = Bu$,

where $B : L^2(\Omega) \rightarrow H_0^1(\Omega)$ is a solution operator of (3).

Using this and (2), it follows

$$(\nabla u, \nabla w)_{L^2} = (g(u), w)_{L^2}, \quad \forall w \in H_0^1(\Omega), \quad (4)$$

where $g = 1/\epsilon^2(f - \delta B) : H_0^1(\Omega) \rightarrow L^2(\Omega)$.

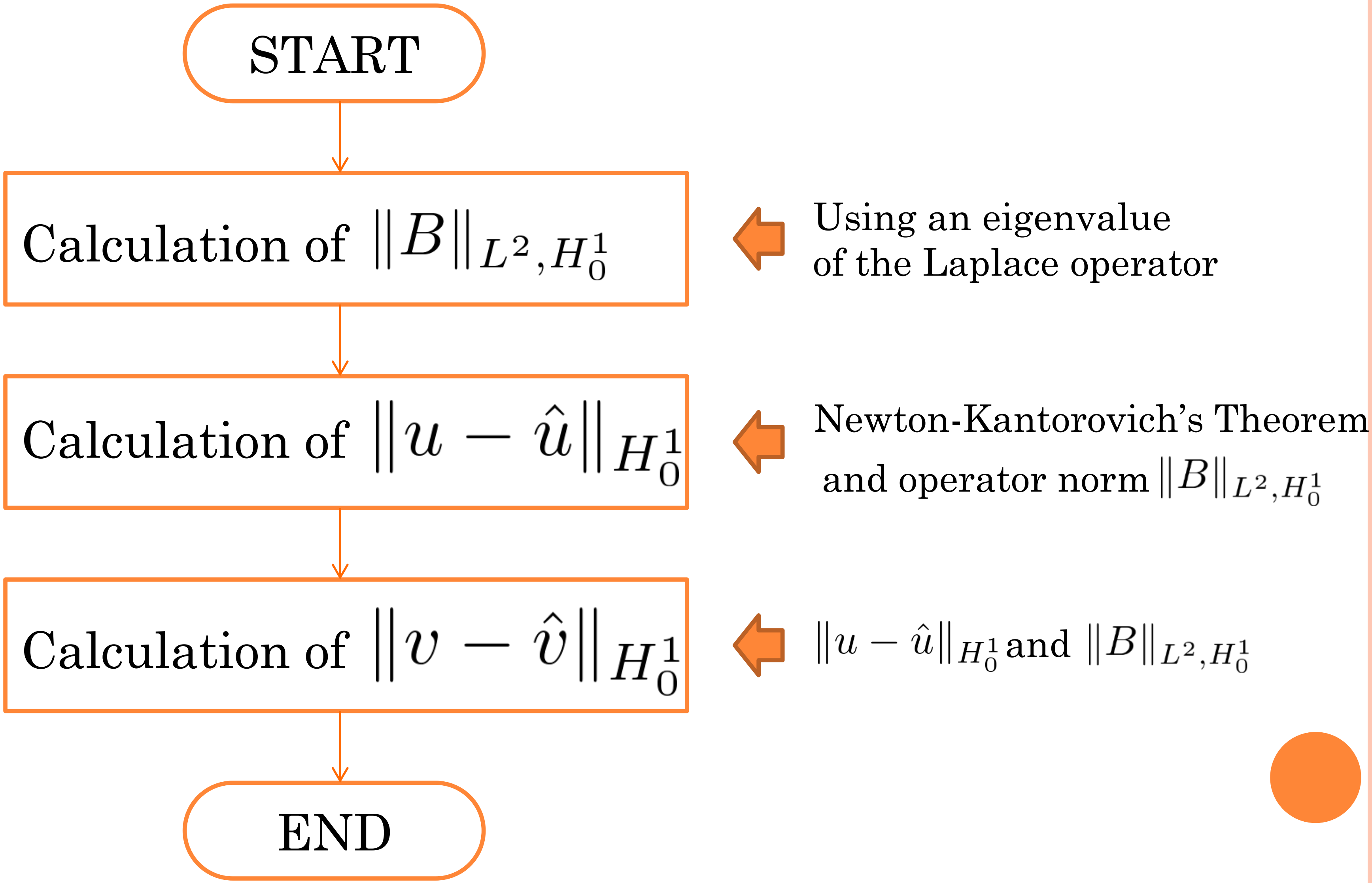
Y. Watanabe has studied this type of equation (3) and (4) by Nakao's theory[1].

[1] Y. Watanabe, A Numerical Verification Method for Two-Coupled Elliptic Partial Differential Equation, Japan Jurnal of Industrial and Applied Mathematics, 26 (2009), pp.233-247

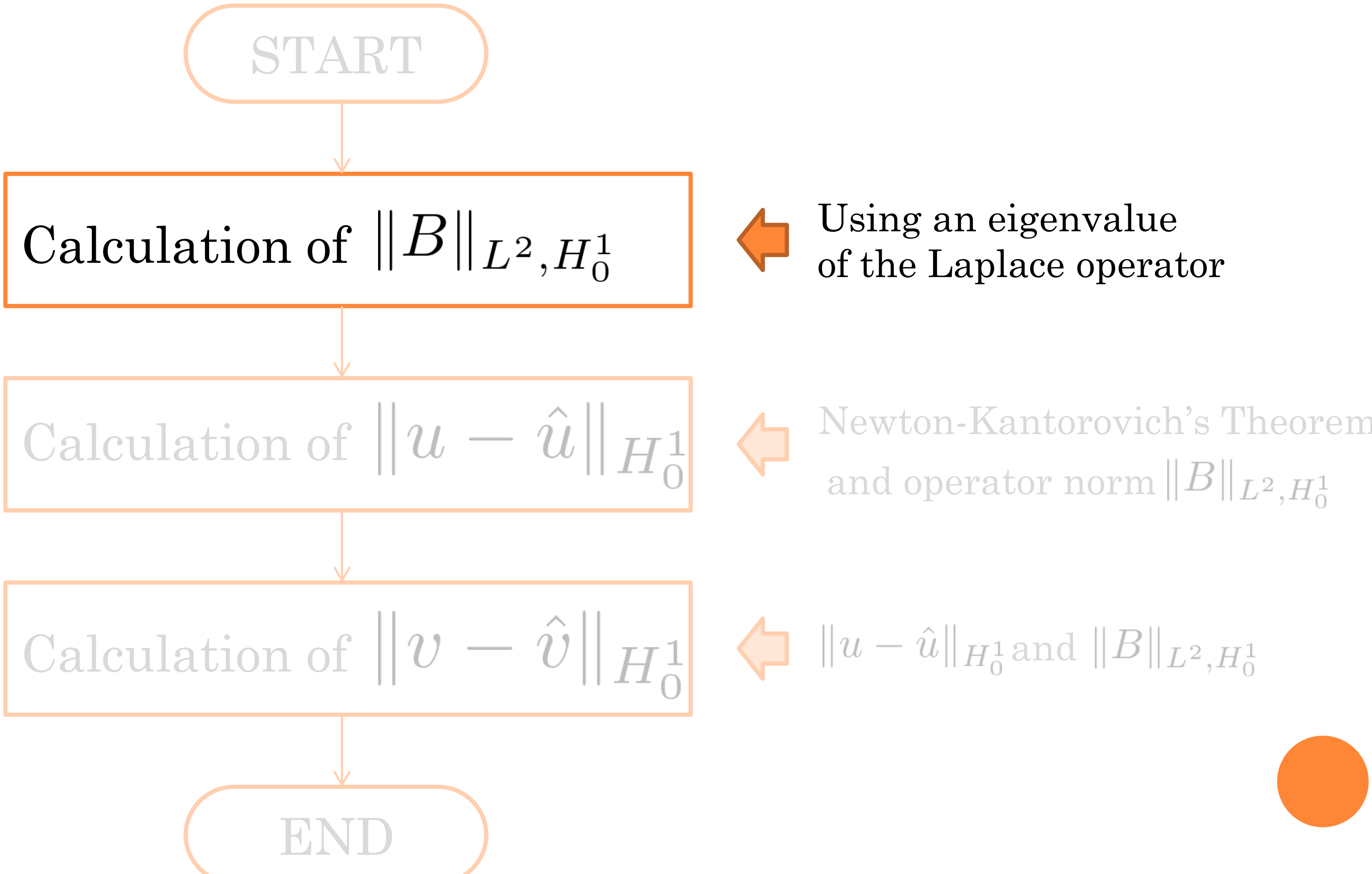
# 2.Purpose

Our purpose is the proof of the uniqueness and existence of the solution for equation (3) and (4) using the Newton-Kantorovich's theorem and the operator norm $\|B\|_{L^2,H_0^1}$ .

# Flowchart for the computer assisted proof



# Flowchart for the computer assisted proof



# 3. OPERATOR $B$ AND ESTIMATION OF THE NORM

Linear operator $\mathcal{L}: H_0^1(\Omega) \to H^{-1}(\Omega)$ and embedding identity operator $\mathcal{I}: L^2(\Omega) \to H^{-1}(\Omega)$ is defined as

$$\begin{aligned} \langle \mathcal{L}v, w \rangle &:= (\nabla v, \nabla w)_{L^2} + \gamma(v, w)_{L^2}, \\ \langle \mathcal{I}v, w \rangle &:= (v, w)_{L^2}, \quad \forall w \in H_0^1(\Omega) \end{aligned}$$

Then, equation (3) transform as following.

$$\text{Find } v \in H_0^1(\Omega), \text{ satisfying } \mathcal{L}v = \mathcal{I}u.$$

# 3. OPERATOR $B$ AND ESTIMATION OF THE NORM

If $\gamma$ is not an eigenvalue of the Laplace operator, there exists the solution operator *B*.
Thus, we define

$$B := \mathcal{L}^{-1}\mathcal{I} : L^2(\Omega) \to H_0^1(\Omega)$$

# 3. Operator $B$ and estimation of the norm

Let $H^{-1}(\Omega)$ be the topological dual space of $H_0^1(\Omega)$. The norm of $\mathcal{T} \in H^{-1}(\Omega)$ is defined as

$$\|\mathcal{T}\|_{H^{-1}} := \sup_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{|\langle \mathcal{T}, v\rangle|}{\|v\|_{H_0^1}}.$$

Let $X$ and $Y$ be Banach space. The set of bounded linear operators is denote by $\mathcal{L}(X,Y)$ with operator norm

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{v \in X\setminus\{0\}} \frac{\|Tv\|_Y}{\|v\|_X}.$$

# 3. Operator $B$ and estimation of the norm

We define the linear operator $\Phi : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ as

$$\langle \Phi v, w \rangle \quad := \quad (\nabla v, \nabla w)_{L^2}, \quad \forall w \in H_0^1(\Omega).$$

As a property of the operator $\Phi$ for all $v \in H_0^1(\Omega)$,

$$\begin{aligned} \|\Phi v\|_{H^{-1}} \quad &= \sup_{w \in H_0^1(\Omega)\backslash\{0\}} \frac{|\langle \Phi v, w \rangle|}{\|w\|_{H_0^1}} \\ &= \sup_{w \in H_0^1(\Omega)\backslash\{0\}} \frac{|(\nabla v, \nabla w)|}{\|w\|_{H_0^1}} = \|v\|_{H_0^1}. \end{aligned}$$

# 3.Operator $B$ and estimation of the norm

We consider an eigenvalue problem:

$$\text{Find } (v, \hat{\lambda}) \in H_0^1(\Omega) \times \mathbb{R}, \text{ s.t. } \mathcal{L}v = \hat{\lambda}\Phi v. \quad (5)$$

Let $K$ be positive real number satisfying

$$K := \max\left\{|\hat{\lambda}|^{-1} : \hat{\lambda} \text{ is satisfied the equation (5)}\right\}$$

# 3.Operator $B$ and estimation of the norm

Then, the operator norm of $\mathcal{L}^{-1}$ is estimated by

$$\begin{aligned} \|\mathcal{L}^{-1}\|_{H^{-1},H_0^1} &= \sup_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\|v\|_{H_0^1}}{\|\mathcal{L}v\|_{H^{-1}}} \\ &= \sup_{v\in H_0^1(\Omega)\setminus\{0\}} \frac{\|\Phi v\|_{H^{-1}}}{\|\mathcal{L}v\|_{H^{-1}}} \\ &\le K. \end{aligned} \tag{6}$$

# 3.OPERATOR $B$ AND ESTIMATION OF THE NORM

We transform a eigenvalue problem (5) into

$$(\nabla v, \nabla w) = -\frac{\gamma}{1-\hat{\lambda}}(v, w),\ \ \forall w \in H_0^1(\Omega).$$

$$\lambda = -\frac{\gamma}{1-\hat{\lambda}}$$

This equation is an eigenvalue problem of the Laplace operator.

A verified evaluation for eigenvalues of the Laplace operator has been shown by X. Liu and S. Oishi[2].

[2] X. Liu and S. Oishi, Verified eigenvalue evaluation for elliptic operator on arbitrary polygonal domain, in preparation

If we get exactly an eigenvalue of the Laplace operator, we have $\hat{\lambda}$. Thus, we can estimate of operator norm of $\mathcal{L}^{-1}$.

# 3.Operator $B$ and estimation of the norm

Let $C_{e,2}$ be the Poincaré constant satisfying

$$\|u\|_{L^2} \le C_{e,2}\|u\|_{H_0^1},\ \forall u \in H_0^1(\Omega).$$

Thus, we have

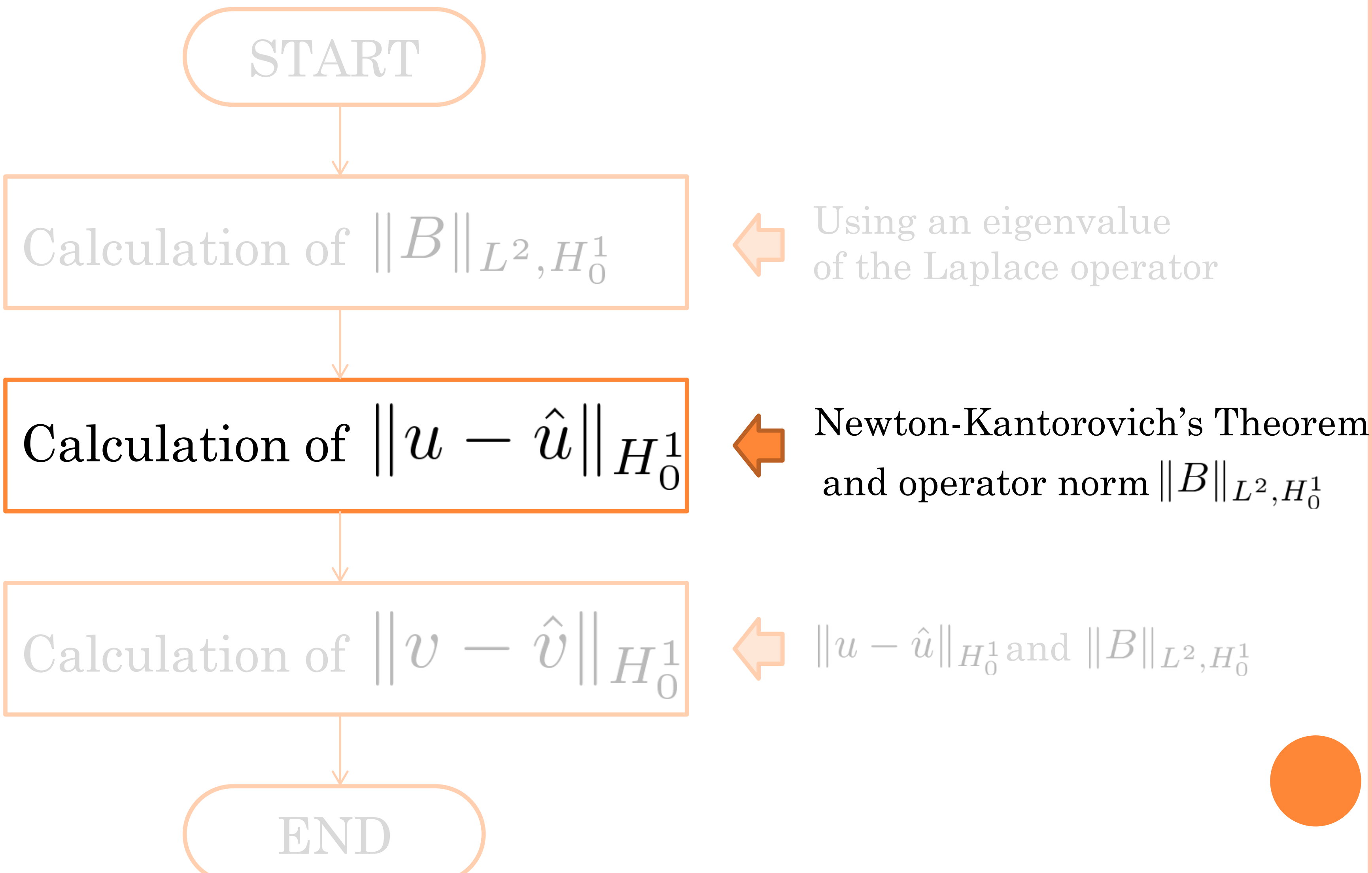
$$\begin{aligned}
\|\mathcal{I}\|_{L^2,H^{-1}} &= \sup_{q\in L^2(\Omega)} \sup_{z\in H_0^1(\Omega)} \frac{|<\mathcal{I}q,z>|}{\|q\|_{L^2}\|z\|_{H_0^1}} \\
&= \sup_{q\in L^2(\Omega)} \sup_{z\in H_0^1(\Omega)} \frac{|(q,z)_{L^2}|}{\|q\|_{L^2}\|z\|_{H_0^1}} \\
&\le C_{e,2} \sup_{z\in H_0^1(\Omega)} \frac{\|z\|_{H_0^1}}{\|z\|_{H_0^1}} = C_{e,2}\,. \quad (7)
\end{aligned}$$

# 3. OPERATOR $B$ AND ESTIMATION OF THE NORM

Using equations (6) and (7), we estimate the operator norm of $B$ as follows:

$$\begin{aligned} \|B\|_{L^2,H_0^1} &= \|\mathcal{L}^{-1}\mathcal{I}\|_{L^2,H_0^1} \\ &\leq \|\mathcal{L}^{-1}\|_{H^{-1},H_0^1}\|\mathcal{I}\|_{L^2,H^{-1}} \\ &\leq C_{e,2}K. \end{aligned}$$

# Flowchart for the computer assisted proof



# 4. Error estimation of an equation (2)

The linear operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ and
The non-linear operator $\mathcal{N} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by

$$\begin{aligned} \langle \mathcal{A}u, w \rangle &:= (\nabla u, \nabla w)_{L^2}, \\ \langle \mathcal{N}(u), w \rangle &:= (g(u), w)_{L^2}, \forall w \in H_0^1(\Omega). \end{aligned}$$

Let $\mathcal{N}'[\hat{u}]$ be a operator on $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ as

$$\langle \mathcal{N}'[\hat{u}]u, w \rangle = (g'[\hat{u}]u, w)_{L^2}, \forall w \in H_0^1(\Omega),$$

where $g'[\hat{u}]$ assume Fréchet different of $g$ on $\hat{u}$ .

# 4. Error estimation of an equation (2)

Let $\mathcal{F}$ be a nonlinear operator on $H_0^1(\Omega)$ into $H^{-1}(\Omega)$.

$$\text{Find } u \in H_0^1(\Omega), \quad \mathcal{F}(u) = \mathcal{A}u - \mathcal{N}(u) = 0 \quad (8)$$

The Fréchet derivative of $\mathcal{F}$ at $\hat{u} \in H_0^1(\Omega)$ denotes

$$\mathcal{F}'[\hat{u}] = \mathcal{A} - \mathcal{N}'[\hat{u}].$$

# 4. Error estimation of an equation (2)

Theorem1 (Newton-Kantorovich's theorem)

Assuming that the Fréchet derivative $\mathcal{F}'[\hat{u}]$ is nonsingular and satisfies

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_{H_0^1} \leq \alpha$$

for the certain positive $\alpha$ .Then, let $\bar{B}(\hat{u}, 2\alpha) := \{v \in H_0^1(\Omega)$ $: \|v - \hat{u}\| \leq 2\alpha\}$ be an closed ball. Let $D \supset \bar{B}(\hat{u}, 2\alpha)$ be an open ball. We assume that for a certain positive $\omega$, the following holds:

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[w] - \mathcal{F}'[m])\|_{H_0^1, H_0^1} \leq \omega\|w - m\|_{H_0^1} \quad w, m \in D$$

If $\alpha\omega < 1/2$ holds, then there is a solution $u \in H_0^1(\Omega)$ of $\mathcal{F}(u) = 0$ satisfying

$$\|u - \hat{u}\|_{H_0^1} \leq \rho = \frac{1-\sqrt{1-2\alpha\omega}}{\omega}$$

Furthermore, the solution $u$ is unique in $\bar{B}(\hat{u}, 2\alpha)$ .

# 4.Error estimation of an equation (2)

We define three constant:

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{H^{-1},H_0^1} \leq C_1$$

$$\|\mathcal{F}(\hat{u})\|_{H^{-1}} \leq C_{2,h}$$

$$\|\mathcal{F}'[w] - \mathcal{F}'[m]\|_{H^{-1}} \leq C_3\|w - m\|$$

We have

$$\alpha := C_1 C_{2,h}$$

$$\omega := C_1 C_3 .$$

If you get three constants $C_1$, $C_{2,h}$, $C_3$, then you can verify Newton-Kantorovich's theorem.

# 4. Error estimation of an equation (2)

Let $\Psi$ be a linear operator on $H_0^1(\Omega)$ into $H^{-1}(\Omega)$,

$$\langle \Psi u, w \rangle \quad := \quad (\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2}, \quad \forall w \in H_0^1(\Omega).$$

Where $\sigma > 0$. We define the $\sigma$-inner product and $\sigma$-norm as

$$(u, w)_\sigma := (\nabla u, \nabla w)_{L^2} + \sigma(u, w)_{L^2},$$

$$\|u\|_\sigma := \sqrt{(u, u)_\sigma}\ .$$

For $u \in H_0^1(\Omega)$, we have following the property.

$$\begin{aligned} \|\Psi u\|_{H^{-1}} &= \sup_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle \Psi u, w \rangle|}{\|w\|_{H_0^1}} \\ &\geq \sup_{w \in H_0^1(\Omega) \setminus \{0\}} \frac{|(u, w)_\sigma|}{\|w\|_\sigma} = \|u\|_\sigma \end{aligned}$$

# 4. Error estimation of an equation (2)

Then, we estimate as

$$\begin{aligned}
\|\mathcal{F}'[\hat{u}]^{-1}\|_{H^{-1},H_0^1} &= \sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{H_0^1}}{\|\mathcal{F}'[\hat{u}]u\|_{H^{-1}}} \\
&= \sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{H_0^1}}{\|\mathcal{A}u-\mathcal{N}'[\hat{u}]u\|_{H^{-1}}} \\
&\le \sup_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{\sigma}}{\|\Psi^{-1}(\mathcal{A}u-\mathcal{N}'[\hat{u}]u)\|_{\sigma}}.
\end{aligned}$$

We denote the calculation of an eigenvalue problem:

Find $(u,\hat{\mu}) \in H_0^1(\Omega)\times\mathbb{R}$

$$(u,w)_\sigma = \hat{\mu}\left(\sigma(u,w)_{L^2} + \frac{1}{\varepsilon^2}(f'[\hat{u}]u,w)_{L^2}\right) - \frac{\delta}{\varepsilon^2}(Bu,w)_{L^2}. \qquad (9)$$

# 4.Error estimation of an equation (2)

Let $V_h$ denote a finite dimensional subspace of $H_0^1(\Omega)$.

An eigenvalue $\hat{\mu}_h$ is satisfying the eigenvalue problem:

Find $(u_h, \hat{\mu}_h) \in V_h \times \mathbb{R}$

$$(u_h, w_h)_\sigma = \hat{\mu}_h \left( \sigma(u_h, w_h)_{L^2} + \frac{1}{\varepsilon^2}(f'[\hat{u}]u_h, w_h)_{L^2} \right) - \frac{\delta}{\varepsilon^2}(Bu_h, w_h)_{L^2},$$

and we wrote $0 < \hat{\mu}_1^h \leq \hat{\mu}_2^h \leq \cdots \leq \hat{\mu}_k^h \leq \cdots$ to eigenvalues.

# 4. Error estimation of an equation (2)

We determine the constant $\sigma$ so that

$$\sigma(u,u)_{L^2} + (g'[\hat{u}]u,u)_{L^2} > 0 .$$

We assume that for a certain positive $K1$, the following holds:

$$\sup_{w \in H_0^1(\Omega)\setminus\{0\}} \frac{|(Bu,w)|}{(u,w)} \leq K_1$$

# 4. Error estimation of an equation (2)

Then,

$$
\begin{aligned}
& \sigma(u,w)_{L^2} + \frac{1}{\varepsilon^2}(f'[\hat{u}]u,w)_{L^2} - \frac{\delta}{\varepsilon^2}(Bu,w)_{L^2} \\
\geq \quad & ((\sigma + \frac{1}{\varepsilon^2}f'[\hat{u}])u,w)_{L^2} - \frac{|\delta|}{\varepsilon^2}\frac{|(Bu,w)|}{(u,w)}(u,w) \\
\geq \quad & ((\sigma + \frac{1}{\varepsilon^2}f'[\hat{u}])u,w)_{L^2} - \frac{|\delta|}{\varepsilon^2}K_1(u,w) \geq 0 \quad .
\end{aligned}
$$

Thus, we have

$$
\sigma \geq -\frac{1}{\varepsilon^2}\left( \operatorname{essinf} \quad f'[\hat{u}] - |\delta|K_1 \right) . \qquad (10)
$$

# 4. Error estimation of an equation (2)

We define the d-inner product and d-norm as

$$(u,w)_d := \sigma(u,w)_{L^2} + (g'[\hat{u}]u, w)_{L^2},$$

$$\|u\|_d = \sqrt{(u,u)_d}\,.$$

Thus,

$$\|u\|_d \le K_2 \|u\|_{L^2}, \qquad (11)$$

where $K_2 := \sqrt{\|\sigma + f'[\hat{u}]\|_{L^\infty} + C_{e,2}^2 \|B\|_{L^2,H_0^1}}\,.$

# 4.Error estimation of an equation (2)

An orthogonal projection $P_{h_\sigma} : H_0^1(\Omega) \to V_h$ is defined by

$$(u - P_{h_\sigma} u, u_h)_\sigma = 0, \ \forall u_h \in V_h \,.$$

There exist positive constants $C_{M_\sigma}$ satisfying

$$\|u - P_{h_\sigma} u\|_\sigma \leq C_{M_\sigma} \| - \Delta u + \sigma u\|_{L^2}$$

for $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Then, we estimate

$$\|u - P_{h_\sigma} u\|_d \quad \leq C_{M_\sigma} K_2 \|u - P_{h_\sigma} u\|_\sigma. \quad (12)$$

# 4. Error estimation of an equation (2)

The following remark is obtained a combination of (10)-(12) and the proof of the Liu-Oishi's theorem[2].

**Remark 1**

If

$$\hat{\mu}_k C_{M_\sigma}^2 K_2^2 < 1,$$

then the eigenvalue $\hat{\mu}_k$ is satisfying

$$\frac{\hat{\mu}_k^h}{\hat{\mu}_k^h C_{M_\sigma} K_2 + 1} \leq \hat{\mu}_k \leq \hat{\mu}_k^h.$$

# 4.Error estimation of an equation (2)

If we get exactly an eigenvalue $\hat{\mu}_k$, we have

$$\|\mathcal{F}'[\hat{u}]^{-1}\|_{H^{-1},H_0^1} \le C_1,$$

where

$$C_1 := \max\left\{\left|\frac{\hat{\mu}_k}{\hat{\mu}_k - 1}\right|\right\}.$$

# 4. Error estimation of an equation (2)

The calculation method of $C_{2,h}$ was proposed by A. Takayasu, X. Liu and S. Oishi[2].

[2]A. Takayasu, X. Liu and S. Oishi, Verified computations to semilinear elliptic boundary value problems on arbitrary polygonal domains, to appear.

$$\|\mathcal{F}(\hat{u})\|_{H^{-1}} \le \|\nabla\hat{u} - p_h\|_{L^2} + C_{M_h}\|g(\hat{u}) - g_h(\hat{u})\|_{L^2} + C_{h,\gamma}C_h\left|\frac{\delta}{\varepsilon}\right|\sqrt{\|B\hat{u}\|_{H_0^1}^2 + \gamma\|B\hat{u}\|_{L^2}^2}$$

Where

- $p_h$ :The smoothing function $p_h$ is defined by the Raviart-Thomas finite element
- $f_h$ : $f_h$ is piecewise-descontinuous on the triangle element, satisfying $(f - f_h, \mu_h)_{L^2} = 0$.

# 4. Error estimation of an equation (2)

$C_L$ is satisfied

$$|((g'[w]-g'[m])u,\psi)| \le C_L \|w-m\|_{H_0^1}\|u\|_{H_0^1}\|\psi\|_{H_0^1}$$

Here, $w, m \in D,\ \ u, \psi \in H_0^1(\Omega)$.

Then, we have

$$\begin{aligned}
&\|\mathcal{F}'[z]-\mathcal{F}'[w]\|_{H_0^1,H^{-1}} \\
&\quad = \|\mathcal{N}'[z]-\mathcal{N}'[w]\|_{H_0^1,H^{-1}} \\
&\quad = \sup_{0\neq\phi\in H_0^1(\Omega)} \frac{\|(\mathcal{N}'[z]-\mathcal{N}'[w])\phi\|_{H^{-1}}}{\|\phi\|_{H_0^1}} \\
&\quad = \sup_{0\neq\phi\in H_0^1(\Omega)} \sup_{0\neq\psi\in H_0^1(\Omega)} \frac{|<(\mathcal{N}'[z]-\mathcal{N}'[w])\phi,\psi>|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}} \\
&\quad = \sup_{0\neq\phi\in H_0^1(\Omega)} \sup_{0\neq\psi\in H_0^1(\Omega)} \frac{|((g'[z]-g'[w])\phi,\psi)|}{\|\phi\|_{H_0^1}\|\psi\|_{H_0^1}}.
\end{aligned}$$

Therefore, one can put $C_3 := C_L$ .

# Flowchart for the computer assisted proof

START

↓

Calculation of $\|B\|_{L^2,H_0^1}$ ⇐ Using an eigenvalue of the Laplace operator

↓

Calculation of $\|u-\hat{u}\|_{H_0^1}$ ⇐ Newton-Kantorovich's Theorem and operator norm $\|B\|_{L^2,H_0^1}$

↓

Calculation of $\|v-\hat{v}\|_{H_0^1}$ ⇐ $\|u-\hat{u}\|_{H_0^1}$ and $\|B\|_{L^2,H_0^1}$

↓

END

# 5.Error estimation of an equation (3)

Let $\hat{u}, \hat{v} \in X_h$ be approximate solutions.

Then, we estimate

$$\|v^* - \hat{v}\|_{H_0^1} \leq C_{e,2} C_h K \rho + K \|\nabla \hat{v} - q_h\|_{L^2}$$

$$+ C_{h,1} K \|k(\hat{v} - k_h(\hat{v}))\|_{L^2}$$

where $k(\hat{v}) = \hat{u} - \gamma \hat{v}$.

# 6. COMPUTATIONAL RESULTS

We would like to consider the system of elliptic partial differential equations:

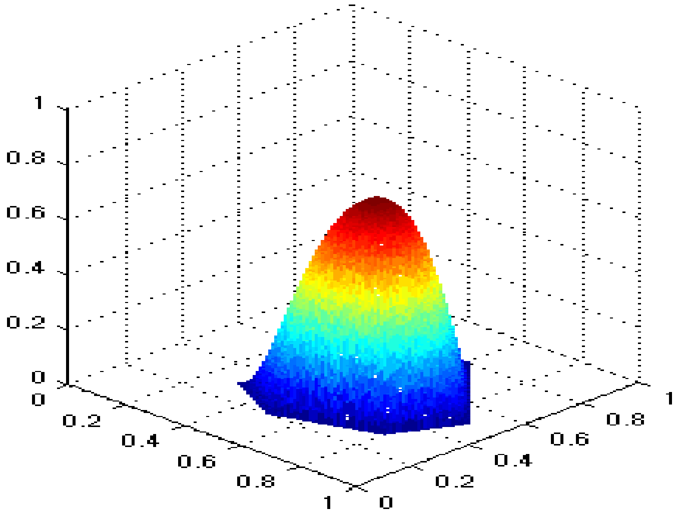
$$\begin{cases} -\varepsilon^2 \Delta u = u - u^3 - \delta v, & \text{in } \Omega, \\ -\Delta v = u - \gamma v, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega \end{cases}$$

where

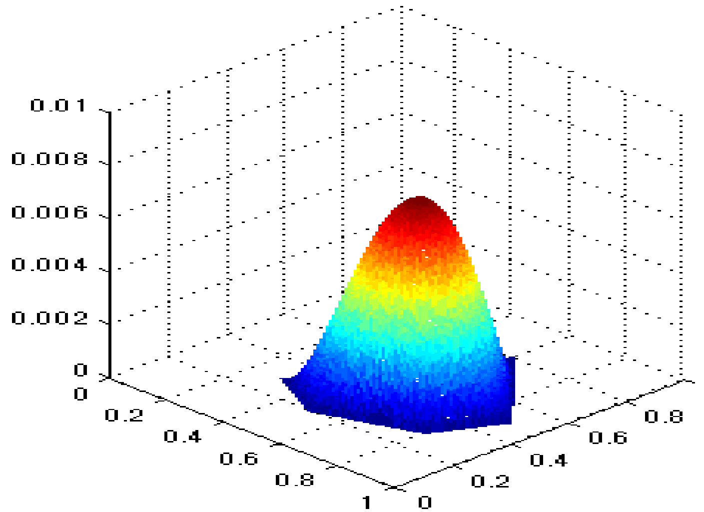
$$\begin{aligned} \varepsilon &= 0.08, \\ \delta &= 0.2, \\ \gamma &= -1.2 \end{aligned}$$

and maximum mesh size $h = 2^{-7}$.

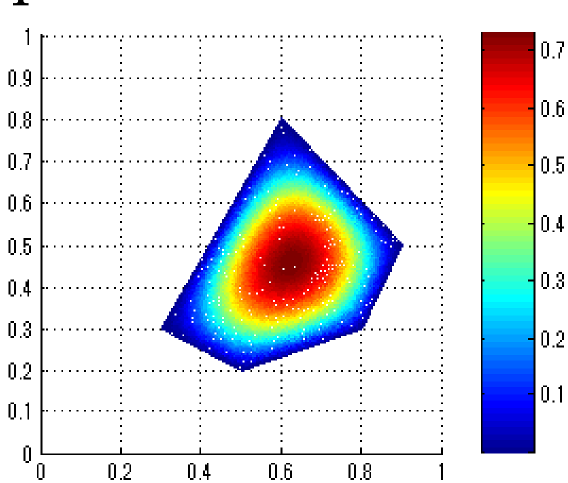
# 6.COMPUTATIONAL RESULTS



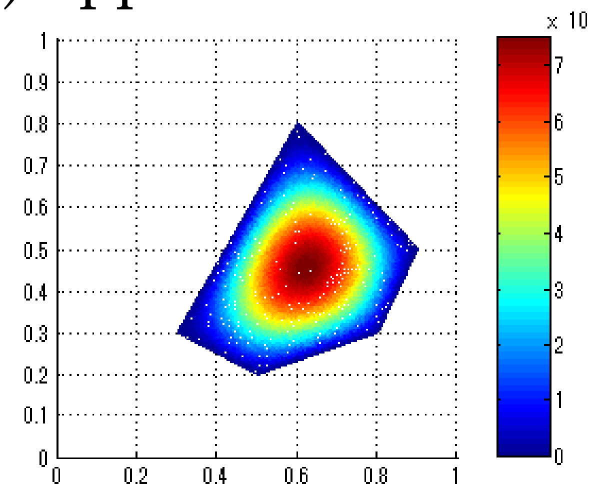
(a) approximate solution $\hat{u}$



(b) approximate solution $\hat{v}$

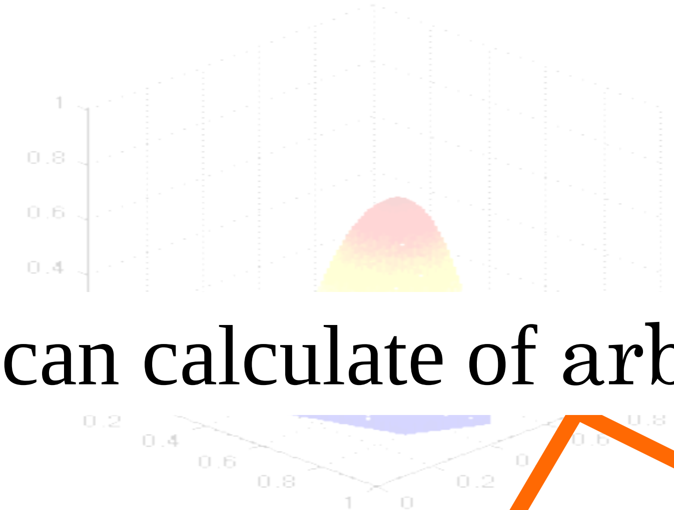


(c) approximate solution $\hat{u}$

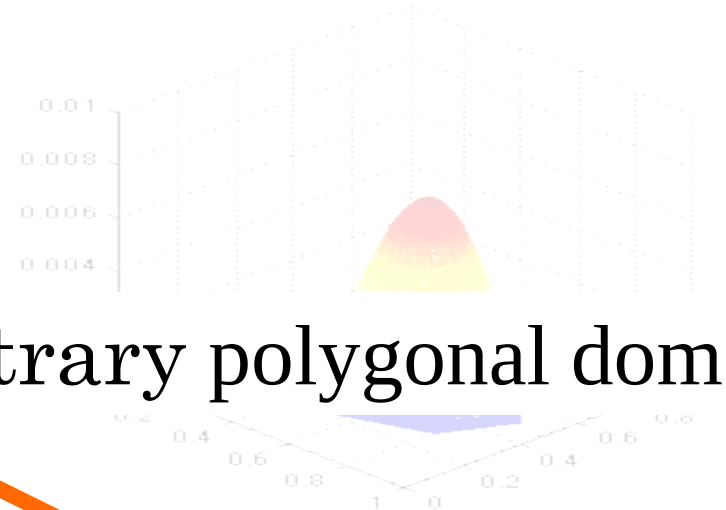


(d) approximate solution $\hat{v}$

# 6.Computational results

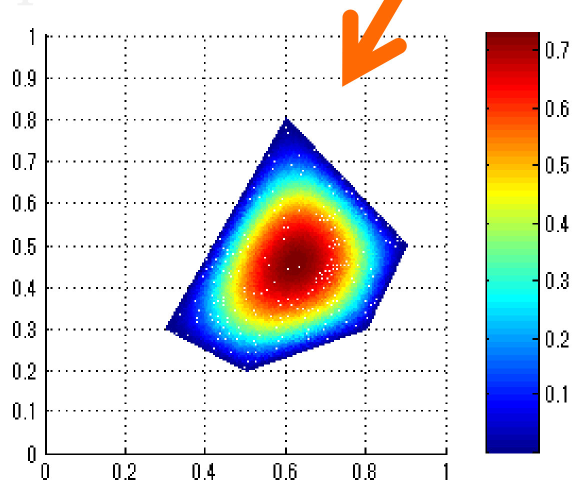



(a) approximate solution $\hat{u}$

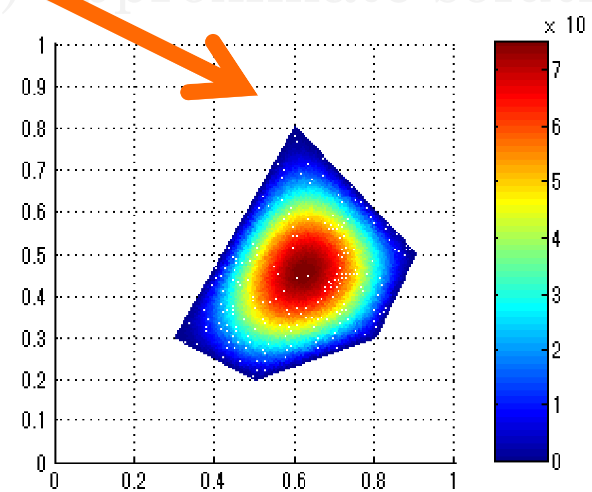



(b) approximate solution $\hat{v}$

We can calculate of arbitrary polygonal domain!!




(c) approximate solution $\hat{u}$




(d) approximate solution $\hat{v}$

# 6. Computational results

Then, we have

$$\|B\|_{L^2,H_0^1} \le 0.097,$$

$$C_1 = 2.3813,$$

$$C_{2,h} = 0.0024,$$

$$C_3 = 32.3119,$$

$$C_1{}^2 \times C_{2,h} \times C_3 = 0.4393 < 0.5,$$

$$\|u - \hat{u}\|_{H_0^1} \le 0.0085,$$

$$\|v - \hat{v}\|_{H_0^1} \le 8.0814 \times 10^{-5}.$$

Therefore, the uniqueness and existence of the local solution is proved.

# Thank you for your attention!!
# Спасибо за ваше внимание!!