Endpoint and Midpoint Interval Representations Theoretical and Computational Comparison

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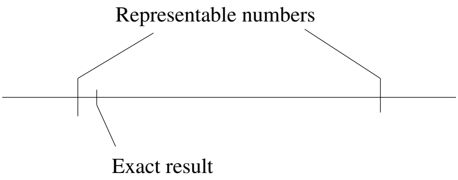
Example

Task: Compute an interval enclosure for $x = 1/15$ based on IEEE standard double precision.



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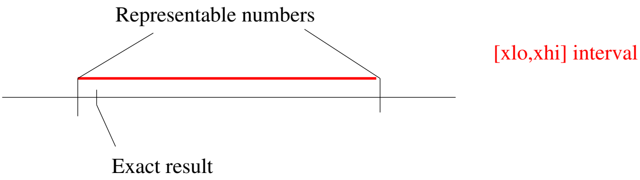

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$x \in [6.6666666666666657415 \times 10^{-2}, 6.6666666666666796193 \times 10^{-2}]$
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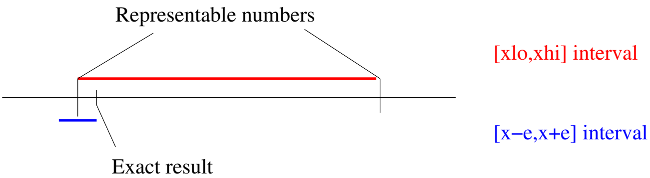

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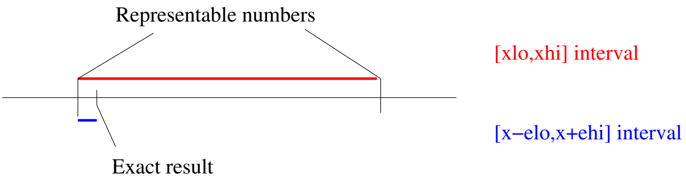

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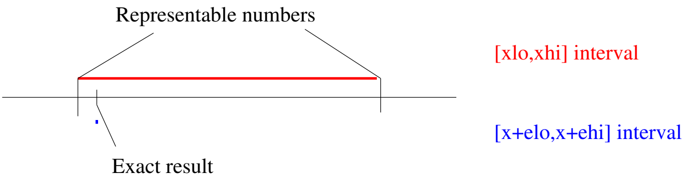

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Width: $< 10^{-30}$

Interval Types

Let Ω be the set of numbers representable in IEEE double precision format

We consider four kinds of intervals:

1. $[x_{lo}, x_{hi}]$ such that $x_{lo}, x_{hi} \in \Omega$
2. $[x - e, x + e]$ such that $x, e \in \Omega$
3. $[x - e_{lo}, x + e_{hi}]$ such that $x, e_{lo}, e_{hi} \in \Omega$; $e_{lo}, e_{hi} \geq 0$
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$\oplus, \otimes$ are *round to nearest, ties to even* addition and multiplication

$\overline{+}, \underline{+}$ denote operations rounded up/down

IEEE Floating Point Format

binary64 number (double precision) stored as:

- 1 bit for sign s
- 11 bit exponent e
- 52 bit mantissa m

It most cases it represents number:

$$(-1)^s \times (1 + 2^{-52} m) \times 2^{(e-1023)}$$

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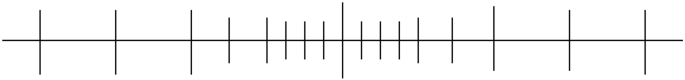

0

$\rightarrow$ If mantissa was 4 bits long, then binary numbers 100100.0 and 0.001001 are exactly representable, but the number 100100.001001 is not representable.

Computing With Midpoint Intervals

Dekker(1971) showed that given $a, b \in \Omega$
$(a \oplus b) - (a + b) \in \Omega$ and $(a \otimes b) - (a \times b) \in \Omega$
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We can then compute $[x_1 - e_1, x_1 + e_1] + [x_2 - e_2, x_2 + e_2]$:
1. $(x, e_3) := \mathtt{add}(x_1, x_2)$
2. $e := e_1 \overline{+} e_2 \overline{+} |e_3|$
3. **return** $[x - e, x + e]$

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Case	$[a_{lo}, a_{hi}]$	$[a - e, a + e]$	$[a + e_{lo}, a + e_{hi}]$
$\|b\| < \epsilon$	2ϵ	$2\|b\| + O(\epsilon^2)$	$O(\epsilon^2)$
$\epsilon \leq \|b\| < 1/2$	2ϵ	$\epsilon + O(\epsilon^2)$	$O(\epsilon^2)$
$1 \leq \|b\|$ and $b < 0$	0	$O(\epsilon^2)$	$O(\epsilon^2)$
$1 \leq \|b\|$ and $b > 0$	2ϵ	$< 2\epsilon + O(\epsilon^2)$	$O(\epsilon^2)$

$\epsilon = 2^{-53}$

Addition With Medium Magnitude Difference

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In intervals of the second kind, we compute $(\texttt{res}, \texttt{err}) := \texttt{add}(a, b)$. The expected magnitude of $\texttt{err}$ is $\epsilon/2$

The expected error introduced is $2\texttt{err} = \epsilon$

Implementation Pitfalls and Wide Intervals

Special care has to be taken for underflowing multiplication

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Multiplication of wide intervals $[1-1, 1+1] \times [1-1, 1+1]$ yields suboptimal results ($[1-3, 1+3]$)

$\rightarrow$ shift of the interval center is required in intervals of the second kind

Computational Experiments

1. Add 10000 numbers with high magnitude difference
2. Add 10000 numbers with moderate magnitude difference
3. Add 10000 numbers with alternating sign
4. Add 10000 numbers with similar magnitude
5. Multiply 10000 numbers

	$[a, b]$	$[a-e, a+e]$	$[a-e_{lo}, a+e_{hi}]$
Test	Interval width		
1	2.2×10^{-12}	1.6×10^{-27}	8.6×10^{-40}
2	2.2×10^{-12}	1.1×10^{-12}	0.0
3	0.0	0.0	0.0
4	2.2×10^{-11}	1.8×10^{-11}	0.0
Mul Narrow	1.7×10^{-12}	1.1×10^{-12}	8.2×10^{-27}
Mul Wide	1.3644389579658	1.7683310177080	1.3644389579634

Arithmetic Operations Count and Timings

	$[a, b]$	$[a-e, a+e]$	$[a-e_{lo}, a+e_{hi}]$
Addition			
Add	2	8	10
Time(10^9 operations)	$37s$	$48s$	$49s$
Multiplication			
Add	0	14	29
Mul	8	9	22
Time(10^9 operations)	$57s$	$63s$	$114s$

Application to Rigorous Polynomial

Possible storage formats:

1. $\sum_i [a_i, b_i] x^i$
2. $(\sum_i a_i x^i) + [-e, e]$
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In second and third case we need less memory to store polynomial

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We have compared three kinds of intervals

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Thank you for you attention.