Limitations of complex interval Gauss-Seidel iterations

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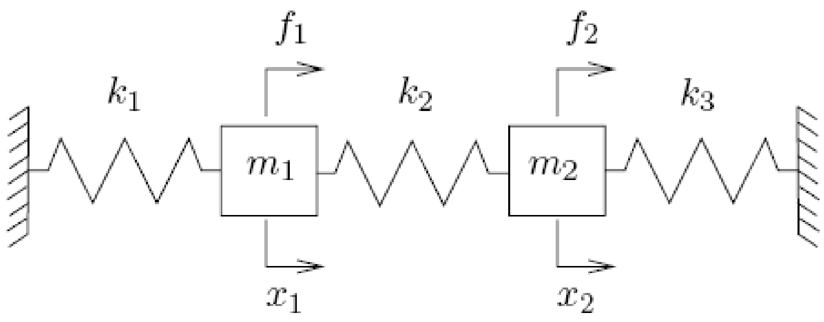
Motivations

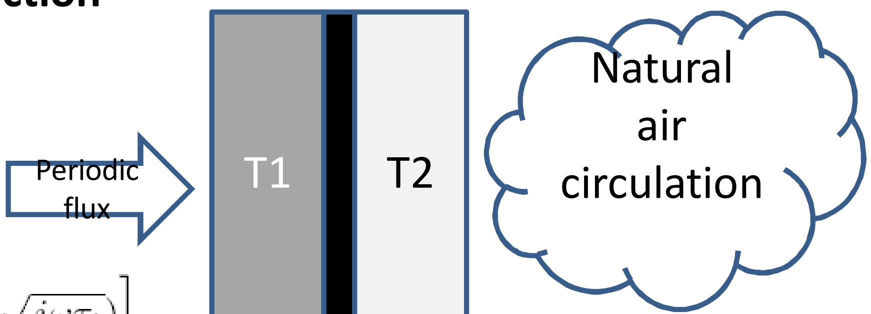


Example 1. Mechanical system with several parameters

$$\left(\begin{bmatrix}(1+i\eta_1)k_1^0+(1+i\eta_2)k_2^0 & -(1+i\eta_2)k_2^0\\ -(1+i\eta_2)k_2^0 & (1+i\eta_2)k_2^0+(1+i\eta_3)k_3^0\end{bmatrix} + e_1\begin{bmatrix}(1+i\eta_1)k_1^1 & 0\\ 0 & 0\end{bmatrix} + e_2 \begin{bmatrix}(1+i\eta_2)k_2^1 & -(1+i\eta_2)k_2^1\\ -(1+i\eta_2)k_2^1 & (1+i\eta_2)k_2^1\end{bmatrix} + e_3\begin{bmatrix}0&0\\0&(1+i\eta_3)k_3^1\end{bmatrix} - \omega^2\begin{bmatrix}m_1&0\\0&m_2\end{bmatrix}\right)\begin{Bmatrix}H_1\\H_2\end{Bmatrix}=\begin{Bmatrix}f_1\\f_2\end{Bmatrix}$$

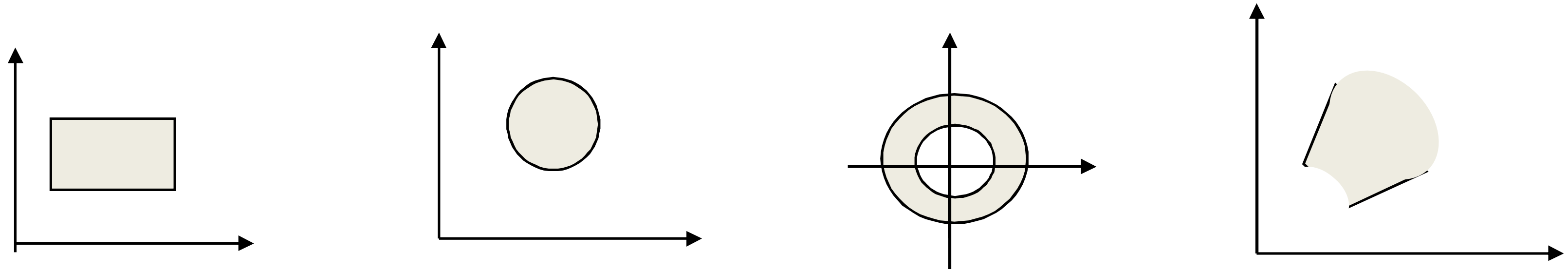
Example 2. Estimation of thermal transfer function

$$F(j\omega) \equiv \frac{T_2(j\omega)}{T_1(j\omega)} = \frac{1}{D_1 + D_2}$$



$$\begin{cases} D_1 = \cosh(\sqrt{j\omega\tau_2})\left[(1+Rh)\cosh(\sqrt{j\omega\tau_1}) + \frac{hR_1}{\sqrt{j\omega\tau_1}}\sinh(\sqrt{j\omega\tau_1})\right] \\ D_2 = \sinh(\sqrt{j\omega\tau_2})\left[\left(\frac{R}{R_2}\sqrt{j\omega\tau_2} + \frac{hR_2}{\sqrt{j\omega\tau_2}}\right)\cosh(\sqrt{j\omega\tau_1}) + \frac{R_1}{R_2}\sqrt{\frac{\tau_2}{\tau_1}}\sinh(\sqrt{j\omega\tau_1})\right] \end{cases}$$

Basic objects



- circular interval – complex set $<c,r> = \{x \in C: |x-c|<r\}, r \geq 0$
- *rad* $<c,r> = r$, *mid* $<c,r> = c$
- **A**x=**b** – system of linear equations
- United solution set: $\Xi_{uni} = \{x \in C^n \mid \exists A \in \mathbf{A}, \exists b \in \mathbf{b}: Ax=b\}$

Disadvantages

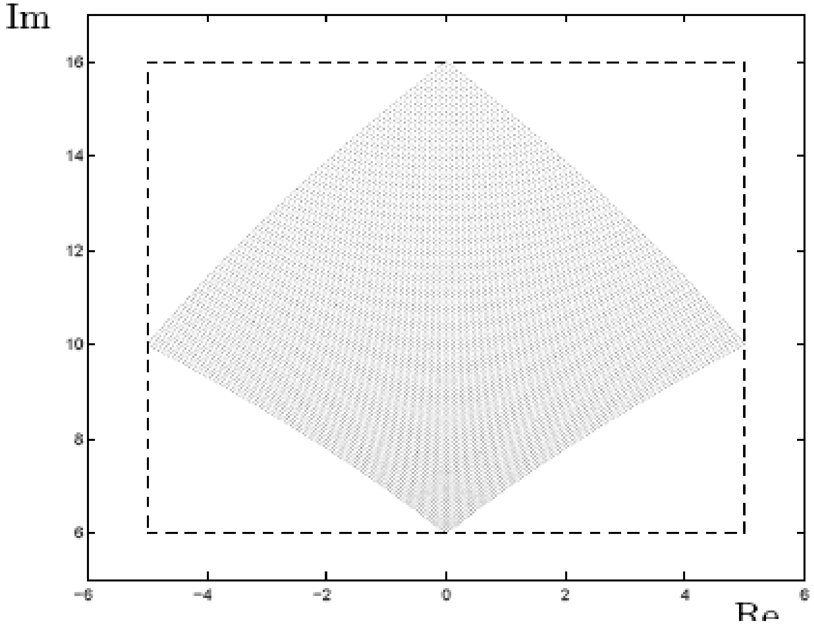
Sector

- Additional operations during addition\subtraction; roughening during this operations.
- Problems with solution sets: Beeck's characterization is wrong. $\Xi(\mathbf{A}, \mathbf{b}) \neq \{x | 0 \epsilon \mathbf{A}x - \mathbf{b}\}$

Rectangular

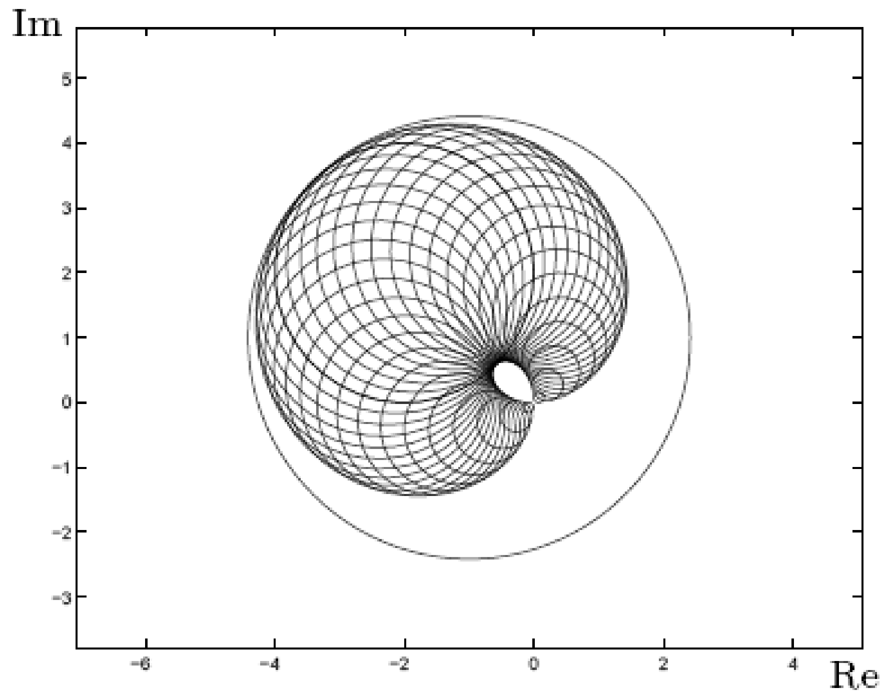
- Double set of interval parameters.
- Arithmetic problems. Multiplication is unassociative

Multiplication problem



[1,2]+[1,2]i × [3,4]+[3,4]i

<1,1> × <-1+i, 1>



Gauss-Seidel algorithm pseudocode

- On input: system ***A***x=***b*****,** an enclosure ***χ*** for Ξ_{uni}
  a stopping criterion ε.

d:= +infty

- DO WHILE (d> ε)

  FOR i=1 TO n

$$\chi_i := x_i \cap (b_i - \sum_{j=1}^{i-1} a_{ij}\chi_j - \sum_{j=i+1}^{n} a_{ij}\chi_j) / a_{ij}$$

  IF $\chi_i = \varnothing$ THEN STOP (No solutions)

  END IF

  END FOR

  d:= dist (χ, ***x***)

  χ := ***x***

END DO

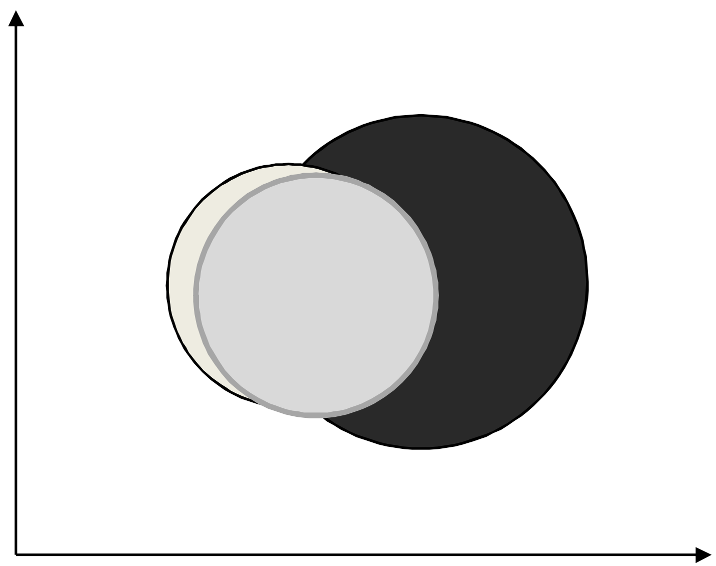
Statement 1

- “Classic” interval Gauss-Seidel method can be generalized for the complex case with replacement of real interval operations by complex ones, and minor corrections in pseudocode.

hull (<a,r>,<b,R>) = <c,P>

$$P = \frac{\sqrt{(r + R + |a - b|)(r + R - |a - b|)}}{2} \cdot \frac{\sqrt{(r - R + |a - b|)(R - r + |a - b|)}}{|a - b|},$$

$$c = \frac{a\sqrt{R^2 - P^2} + b\sqrt{r^2 - P^2}}{\sqrt{R^2 - P^2} + \sqrt{r^2 - P^2}}$$



Limitations

Theorem 1. Complex Gauss-Seidel iteration method do not deteriorate outer estimation of solutions set at any step (but not necessarily converge to optimal outer estimate).

Real case: if **A** is not an H-matrix, then there exist “improvement-resistant” starting estimates of any width for the system **A**x = 0.

Complex trace domination

We will call a circular interval $n \times n$ matrix *circular trace dominant matrix* (CTD-matrix), if, for every non-zero interval *n*-vector **u** with *mid* $(\mathbf{u}_i) = 0$, the condition

$$\left| \sum_{i \neq j} \boldsymbol{a}_{ij} \boldsymbol{u}_j \right| < |\boldsymbol{a}_{ii} \boldsymbol{u}_i| \quad \text{is true for every } i$$

Inflation process: mid $\mathbf{v}_i$=mid $\mathbf{u}_i$, rad $\mathbf{v}_i = c \cdot$rad $\mathbf{u}_i$ $(c>1)$

Inflation process

Circular operations:

Multiplication: $\langle a, r\rangle \cdot \langle b, R\rangle = \langle ab, |a|R + |b|r + Rr\rangle$

Inversion: $\frac{1}{\langle a, r\rangle} = \langle\frac{a^*}{|a|^2 - r^2}, \frac{r}{|a|^2 - r^2}\rangle$

here is *a** complex conjugate element for *a*

$$\langle ab, crR\rangle \subset \langle a, r\rangle \cdot \langle b, cR\rangle$$

$$\langle a, r\rangle \subset \langle b, R\rangle \Leftrightarrow |b - a| \leq R - r$$

Complex case limitations

- **Theorem 2**. If, in the system of equations **A**x = 0, the matrix **A** is not an CTD-matrix, then there exits a starting enclosure **x** of any width that cannot be improved by Gauss-Seidel iterations use.

- Strong difference: $\exists \boldsymbol{u}, \boldsymbol{u} \neq \boldsymbol{0}, mid(\boldsymbol{u_i}) = \boldsymbol{0}, i = 1 \ldots n$

$$\left| \sum_{i \neq j} \boldsymbol{a}_{ij} \boldsymbol{u}_j \right| \geq \lambda |\boldsymbol{a}_{ii} \boldsymbol{u}_i|$$

Complex case limitations (cont.)

- **Statement 2.** If the coefficient λ is large enough, then the result can be generalized for non-zero **b**

$$\max_{i=1..n}\left\{\frac{(|mid(\boldsymbol{a}_{ii})|+\boldsymbol{r}_{ii}^{2})}{mid^{2}(\boldsymbol{a}_{ii})-\boldsymbol{r}_{ii}^{2}}\sum_{j\neq i}|\boldsymbol{a}_{ij}|\right\}$$

Statement 3. Class of CTD-matrices is empty

Conclusion

- Generalization of interval Gauss-Seidel algorithm for the complex case is possible.
- Very strong requirement of CTD-matrix is essentially narrowing the applicability of the Gauss-Seidel iterations. In fact, the class of CTD-matrices is empty.
- There still exists a certain class of matrices close to CTD-matrices that the application of the Gauss-Seidel complex interval method for them produces good results.

Sources

- Example with mechanical system - Y. Candau, T. Raissi, N. Ramdani and L. Ibos, “Analysis of Mechanical Systems using Interval Computations applied to Finite Elements Methods”, Journal of Sound and Vibration.
- Examle with thermal transfer – same authors, “Complex interval arithmetic using polar form” Reliable Computing.
- Examples on multiplication problem – quoted from the book of S.P. Shary “Finite Interval Analysis”.

Thank you for your attention!