

Comparisons of implementations of Rohn's modification in PPS-methods for interval linear systems

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Solution set

The **united solution set** of the interval linear system is the set

$$\Xi(\mathbf{A}, \mathbf{b}) = \{ x \in \mathbb{R}^n \mid (\exists A \in \mathbf{A})(\exists b \in \mathbf{b})(Ax = b) \},$$

formed by solutions to all the point systems $Ax = b$ with $A \in \mathbf{A}$ and $b \in \mathbf{b}$.

An interval matrix \mathbf{A} is assumed to be nonsingular, that is, to contain nonsingular point matrices. So, the solution set $\Xi(\mathbf{A}, \mathbf{b})$ of the interval linear system is bounded.

The problem of optimal outer estimation of the solution set to interval linear system:

Find an interval vector $\mathbf{U} \subset \mathbb{IR}^n$, that has the least possible width and contains the solution set $\Xi(\mathbf{A}, \mathbf{b})$ of the interval linear system $\mathbf{A}x = \mathbf{b}$;

or in componentwise form:

find $\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$ and $\max\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$, $\nu = 1, 2, \dots, n$,
or its most precise estimates from below and from above respectively.

We confine ourselves to computing only $\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$, since

$$\max\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\} = -\min\{x_\nu \mid x \in \Xi(\mathbf{A}, -\mathbf{b})\}$$

for fixed ν .

Parameter partitioning methods

The main idea of parameter partitioning method

is refine sequentially outer componentwise estimates of the solution set to interval linear system $\mathbf{A}x = \mathbf{b}$ by means of subdividing interval elements of the matrix \mathbf{A} and right-hand side vector \mathbf{b} .

The solving of this problem is simplified if we take into consideration the following result:

Beck-Nickel theorem. *If \mathbf{A} is regular, then for any $\nu \in \{1, 2, \dots, n\}$ exact componentwise estimates of the points from the solution set,*

$$\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}, \max\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$$

*are attained at the so-called **extreme** matrices and right hand-side vectors made up of the **endpoints** of \mathbf{A} and \mathbf{b} .*

Notation:

$Encl$ a fixed method that computes an enclosure of the solution set to ILAS (we shall call it *basic method*);

$Encl(\mathbf{A}, \mathbf{b}) \in \mathbb{IR}^n$ an interval enclosure, produced by the method $Encl$, of the solution set to ILAS $\mathbf{A}x = \mathbf{b}$, i.e., $Encl(\mathbf{A}, \mathbf{b}) \supseteq \Xi(\mathbf{A}, \mathbf{b})$;

$\Upsilon(\mathbf{A}, \mathbf{b})$ the lower endpoint of the ν -th component ($\nu = 1, 2, \dots, n$) of the interval enclosure $Encl(\mathbf{A}, \mathbf{b})$, i.e., $\Upsilon(\mathbf{A}, \mathbf{b}) := \underline{Encl(\mathbf{A}, \mathbf{b})}_\nu$;

\mathbf{A}' и \mathbf{A}'' matrices obtained from \mathbf{A} through replacing the element \mathbf{a}_{ij} by the endpoints $\underline{\mathbf{a}}_{ij}$ and $\overline{\mathbf{a}}_{ij}$ respectively;

\mathbf{b}' и \mathbf{b}'' vectors obtained from \mathbf{b} through replacing the element \mathbf{b}_i by the endpoints $\underline{\mathbf{b}}_i$ and $\overline{\mathbf{b}}_i$ respectively.

If the estimate $\Upsilon(\mathbf{A}, \mathbf{b})$ is inclusion monotone with respect to \mathbf{A} and \mathbf{b} , then having solved the two interval "systems-descendants" $\mathbf{A}'x = \mathbf{b}'$ and $\mathbf{A}''x = \mathbf{b}''$, we can get **better estimate** for $\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$ from below as

$$\min\{\Upsilon(\mathbf{A}', \mathbf{b}'), \Upsilon(\mathbf{A}'', \mathbf{b}'')\}.$$

Parameter partitioning method

consists in sequential refining the estimate of $\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$ by means of partitioning the system $\mathbf{A}x = \mathbf{b}$ into two systems-descendants by breaking up to the endpoints an interval element of the matrix \mathbf{A} or the vector \mathbf{b} .

We arrange the iterative procedure of the refining the estimate in accordance with well known "branch-and-bound" method. For the natural stopping of the algorithm it is required to reach the complete deintervalization of ILAS or the estimates with accuracy lower than a small quantity ε .

The algorithm is

- **adaptive**, i.e., it considerably uses the information obtained at the preceding steps of the algorithm;
- **sequentially guaranteeing**, i.e., when executed, it generates a sequence of approximate estimates $\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\}$ from below.

Rohn's methodology

For any square matrix \mathbf{A} , the optimal outer estimates of the solution set $\Xi(\mathbf{A}, \mathbf{b})$ are reached at the set of no more than 2^n **extreme solutions** of Oettli-Prager equation

$$|(\text{mid } \mathbf{A})x - \text{mid } \mathbf{b}| = \text{rad } \mathbf{A} \cdot |x| + \text{rad } \mathbf{b}.$$

Let \mathcal{E} be a set of n -vectors with components ± 1 . For fixed $\sigma, \tau \in \mathcal{E}$, the matrices $T_\sigma, A^{\sigma\tau} = \{a_{ij}^{\sigma\tau}\}$ and the vector $b^\sigma = \{b_i^\sigma\}$ are defined as

$$T_\sigma = \text{diag } \{\sigma_1, \dots, \sigma_n\},$$

$$a_{ij}^{\sigma\tau} = \begin{cases} \bar{\mathbf{a}}_{ij}, & \text{if } \sigma_i \tau_j = -1, \\ \underline{\mathbf{a}}_{ij}, & \text{if } \sigma_i \tau_j = 1, \end{cases} \quad b_i^\sigma = \begin{cases} \bar{\mathbf{b}}_i, & \text{if } \sigma_i = 1, \\ \underline{\mathbf{b}}_i, & \text{if } \sigma_i = -1. \end{cases}$$

Rohn theorem about extreme solutions. Let $n \times n$ -matrix \mathbf{A} be nonsingular and \mathbf{b} be an interval n -vector. Then, for every $\sigma \in \mathcal{E}$, the equation

$$\text{mid } \mathbf{A} \cdot x - T_\sigma \cdot \text{rad } \mathbf{A} \cdot |x| = b^\sigma$$

has a unique solution x^σ within $\Xi(\mathbf{A}, \mathbf{b})$ and there holds

$$\text{conv } \Xi(\mathbf{A}, \mathbf{b}) = \text{conv } \{ x^\sigma \mid \sigma \in \mathcal{E} \}.$$

Rohn's method

consists in getting the optimal estimate of the solution set by means of computing all extreme solutions and comparing them.

The algorithm is

- **passive**;
- **finally guaranteeing**, i.e., it computes required optimal outer estimates only after its natural stopping.

Rohn's modification in PPS-algorithm

It follows from Rohn theorem that, for nonsingular matrix \mathbf{A} , extreme componentwise values for the points from the solution set are reached at the set of 4^n matrices $A^{\sigma\tau}$ and associated vectors b^σ

$$\min\{x_\nu \mid x \in \Xi(\mathbf{A}, \mathbf{b})\} = \min_{\sigma, \tau \in \mathcal{E}} ((A^{\sigma\tau})^{-1} b^\sigma)_\nu, \quad \nu = 1, 2, \dots, n.$$

While partitioning the parameters we look at the endpoints of the subdividing interval elements of the matrix and right-hand side vector. In doing this, we connect with every interval system $\mathbf{Q}x = \mathbf{r}$, produced at every step of the algorithm, a **check matrix** $W = \{w_{ij}\}$ and **check vectors** $s = \{s_i\}$ and $t = \{t_j\}$ such that

$$w_{ij} = \begin{cases} -1, & \text{if } \mathbf{q}_{ij} = \bar{\mathbf{a}}_{ij}, \\ 0, & \text{if } \mathbf{q}_{ij} = \mathbf{a}_{ij}, \\ 1, & \text{if } \mathbf{q}_{ij} = \underline{\mathbf{a}}_{ij}, \end{cases} \quad s_i = \begin{cases} -1, & \text{if } \mathbf{r}_i = \underline{\mathbf{b}}_i, \\ 0, & \text{if } \mathbf{r}_i = \mathbf{b}_i, \\ 1, & \text{if } \mathbf{r}_i = \bar{\mathbf{b}}_i. \end{cases}$$

$$w_{ij} = s_i t_j, \quad i, j = 1, 2, \dots, n.$$

The procedure of PPS-algorithm with Rohn's modification

is performed subject to values of the check matrix W and vector s . When subdividing an element q_{kl} of the matrix Q (an element r_k of the right-hand side vector r),

- if $w_{kl} = 0$ ($s_k = 0$), then two systems-descendants $Q'x = r'$ and $Q''x = r''$ are begotten;
- if $w_{kl} = \pm 1$ ($s_k = \pm 1$), then only one descendant is begotten, depending on the sign of w_{kl} (s_k).

After partitioning the leading ILAS, the check matrices and vectors for the systems-descendants are calculated. If at least one object of the triple (W', s', t') is changed, the two remaining ones are recalculated according to the equalities $w_{ij} = s_i t_j, i, j = 1, 2, \dots, n$.

This procedure is being performed until the changes of (W', s', t') stop.

Monotonicity test

Interval extensions of the derivatives of the ν -th component of the solution x to the system $Qx = r$ with respect to elements of the matrix $Q \in \mathbf{Q}$ and the vector $r \in \mathbf{r}$:

$$\frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial q_{ij}} = -\mathbf{y}_{\nu i} \mathbf{x}_j, \quad \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial r_i} = \mathbf{y}_{\nu i},$$

where $\mathbf{Y} = (\mathbf{y}_{\nu i}) \supseteq \{Q^{-1} \mid Q \in \mathbf{Q}\}$ is "inverse interval matrix" for \mathbf{Q} .

If $\tilde{\mathbf{Q}} = (\tilde{q}_{ij})$ and $\tilde{\mathbf{r}} = (\tilde{r}_i)$ are formed of the elements

$$\tilde{q}_{ij} = \begin{cases} [\underline{q}_{ij}, \underline{q}_{ij}], & \text{if } \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial q_{ij}} \geq 0, \\ [\bar{q}_{ij}, \bar{q}_{ij}], & \text{if } \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial q_{ij}} \leq 0, \\ \mathbf{q}_{ij}, & \text{if } \text{int} \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial q_{ij}} \ni 0, \end{cases} \quad \tilde{r}_i = \begin{cases} [\underline{r}_i, \underline{r}_i], & \text{if } \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial r_i} \geq 0, \\ [\bar{r}_i, \bar{r}_i], & \text{if } \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial r_i} \leq 0, \\ \mathbf{r}_i, & \text{if } \text{int} \frac{\partial x_\nu(\mathbf{Q}, \mathbf{r})}{\partial r_i} \ni 0, \end{cases}$$

then $\min\{x_\nu \mid x \in \Xi(\tilde{\mathbf{Q}}, \tilde{\mathbf{r}})\} = \min\{x_\nu \mid x \in \Xi(\mathbf{Q}, \mathbf{r})\}$.

Monotonicity test

is useful to perform before the partitioning of the leading ILAS, which results in deintervalization of some interval elements of its matrix.

Basic methods for enclosing

At every step of the algorithm we use the method (called a basic one) that computes an enclosure of the solution set.

We used the following basic methods:

- Krawczyk method,
- modified Krawczyk method with epsilon-inflation,
- interval Gauss method,
- interval Gauss-Seidel method,
- Hansen-Bleeck-Rohn procedure.

As a basic algorithm, we also used the procedure `verifylss` from INTLAB.

Structure of the working list and its processing

We also analysed the modifications of PPS-methods that implemented various structure and ways of processing of the working list \mathcal{L} , in which the results of the partitioning of the interval linear system are stored:

- 1 the list \mathcal{L} is formed as an unordered list of records (a heap);
- 2 the records of the list \mathcal{L} are in ascending ordered with respect to the estimate $\Upsilon(\mathbf{Q}, \mathbf{r})$;
- 3 in the list \mathcal{L} , the ordered sublist \mathcal{L}_l of the active records, which has the fixed maximal length, is separated, and the rest records are stored as a heap;
- 4 Pankov's method, in which a threshold constant γ is defined and the ordered sublist \mathcal{L}_γ of the active records is separated, for which $\Upsilon(\mathbf{Q}, \mathbf{r}) < \gamma$; the complement $\mathcal{L} \setminus \mathcal{L}_\gamma$ is stored as a heap.

If, during the algorithm run the subset of active records \mathcal{L}_l or \mathcal{L}_γ becomes empty, then a new ordered subset is again formed from the list \mathcal{L} . The threshold constant γ is recalculated for the subset \mathcal{L}_γ .

Algorithms implementation

The implementation of the introduced algorithms are done in **Matlab** with the interval toolbox **INTLAB**.

In addition to the main scheme of PPS-method, we implemented its **modifications**, that used:

- 1 Rohn's technique for eliminating unpromising vertex combinations;
- 2 monotonicity test, with respect to the components of the matrix and the right-hand side vector of the system;
- 3 various enclosure methods for interval linear systems;
- 4 various ways of processing of the working list, in which the results of the partitioning of the interval linear system are stored.

Test interval systems

Example 1

Neumaier's interval linear system:

$$\begin{pmatrix} \theta & [0, 2] & \cdots & [0, 2] \\ [0, 2] & \theta & \cdots & [0, 2] \\ \vdots & \vdots & \ddots & \vdots \\ [0, 2] & [0, 2] & \cdots & \theta \end{pmatrix} x = \begin{pmatrix} [-1, 1] \\ [-1, 1] \\ \vdots \\ [-1, 1] \end{pmatrix}$$

where θ is nonnegative real parameter. The matrix of Neumaier's system of even order n is nonsingular if $\theta > n$, and the one of odd order n is nonsingular if $\theta > \sqrt{n^2 - 1}$. While approaching θ to boundaries of nonsingularity the size of united solution set increases infinitely.

Example 2

Shary's interval linear system:

$$\begin{pmatrix} [n-1, N] & [\alpha-1, 1-\beta] & \cdots & [\alpha-1, 1-\beta] \\ [\alpha-1, 1-\beta] & [n-1, N] & \cdots & [\alpha-1, 1-\beta] \\ \vdots & \vdots & \ddots & \vdots \\ [\alpha-1, 1-\beta] & [\alpha-1, 1-\beta] & \cdots & [n-1, N] \end{pmatrix} x = \begin{pmatrix} [1-n, n-1] \\ [1-n, n-1] \\ \vdots \\ [1-n, n-1] \end{pmatrix}$$

where n – dimension of the system ($n \geq 2$), $0 < \alpha \leq \beta \leq 1$, N – real number such that $N \geq n - 1$. While β decreases, approaching zero, the matrix of the system approaches to a singular one, and the solution set infinitely increases. By means of varying relation between α and β , we can modify the form of the solution set. The optimal componentwise estimates of the solution set \tilde{X} are

$$\min\{x_i \mid x \in \tilde{X}\} = -1/\alpha,$$

$$\max\{x_i \mid x \in \tilde{X}\} = 1/\alpha, \quad i = 1, 2, \dots, n,$$

and they don't depend on N .

Example 3

The interval linear system $\mathbf{A}x = \mathbf{b}$ with the matrix

$$\mathbf{A} = \begin{pmatrix} [1 - r, 1 + r] & 0 & \cdots & 0 & [1 - r, 1 + r] \\ 0 & [1 - r, 1 + r] & \cdots & 0 & [2 - r, 2 + r] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [1 - r, 1 + r] & [2 - r, 2 + r] & \cdots & [n - 1 - r, n - 1 + r] & [n - r, n + r] \end{pmatrix},$$

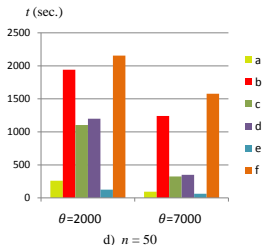
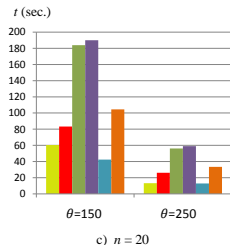
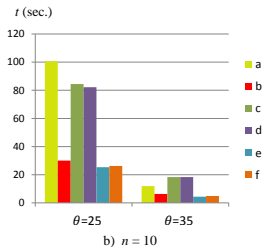
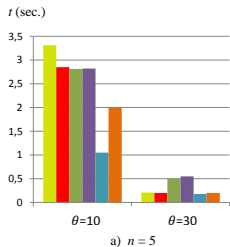
and the right-hand side vector

$$\mathbf{b} = \begin{pmatrix} [1 - R, 1 + R] \\ [1 - R, 1 + R] \\ \vdots \\ [1 - R, 1 + R] \end{pmatrix},$$

where r, R — positive real numbers.

Results of numerical experiments

Example 1. Neumaier's interval linear system.



Basic algorithms:

- a** – Krawczyk method,
- b** – modified Krawczyk method,
- c** – interval Gauss method,
- d** – interval Gauss-Seidel method,
- e** – Hansen-Bleeck-Rohn procedure,
- f** – verifylss.

The characteristics of the properties of ILAS matrix

The spectral radius ρ of the matrix $|(\text{mid } \mathbf{A})^{-1}| \cdot \text{rad } \mathbf{A}$:

$$\rho = \rho(|(\text{mid } \mathbf{A})^{-1}| \cdot \text{rad } \mathbf{A}),$$

and the difference $\Delta\sigma$ between the least and the largest singular values of the matrices $\text{mid } \mathbf{A}$ and $\text{rad } \mathbf{A}$:

$$\Delta\sigma = \sigma_{\min}(\text{mid } \mathbf{A}) - \sigma_{\max}(\text{rad } \mathbf{A}).$$

Ris-Beeck criterion. *Let an interval $n \times n$ -matrix \mathbf{A} be such that $\text{mid } \mathbf{A}$ is nonsingular and*

$$\rho(|(\text{mid } \mathbf{A})^{-1}| \cdot \text{rad } \mathbf{A}) < 1.$$

Then \mathbf{A} is nonsingular.

Rump criterion. *Let an interval $n \times n$ -matrix \mathbf{A} be such that*

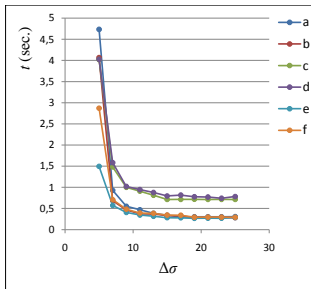
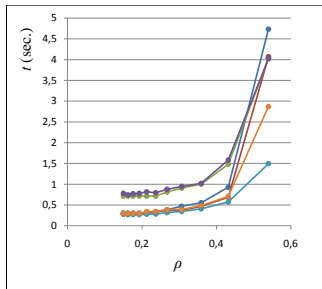
$$\sigma_{\max}(\text{rad } \mathbf{A}) < \sigma_{\min}(\text{mid } \mathbf{A}),$$

then \mathbf{A} is nonsingular.

Influence of the properties of ILAS matrix

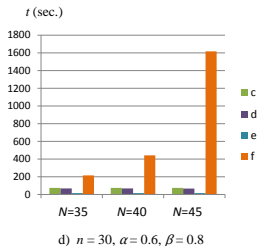
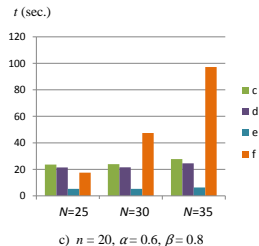
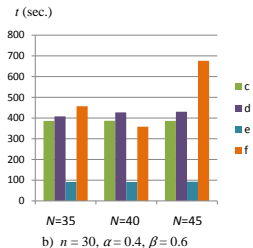
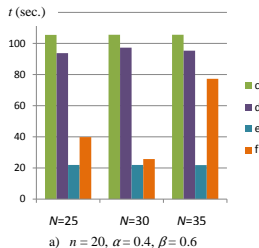
For near-zero ρ and sufficiently large $\Delta\sigma$ the run time of the program is small. But it exponentially increases for the matrices near the boundaries of singularity, i. e. if $\rho \rightarrow 1$ and $\Delta\sigma \rightarrow 0$.

The run time dependence on characteristics ρ and $\Delta\sigma$ of Neumaier's system for the various basic algorithms follows



- a** – Krawczyk method,
- b** – modified Krawczyk method,
- c** – interval Gauss method ,
- d** – interval Gauss-Seidel method,
- e** – Hansen-Bleek-Rohn procedure,
- f** – verifylss

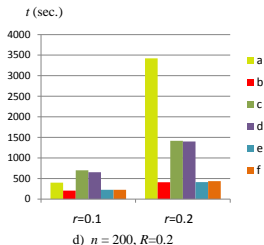
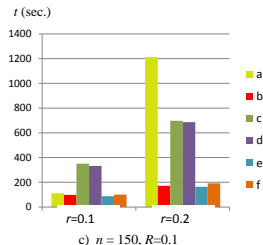
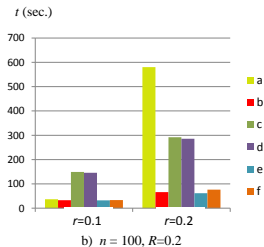
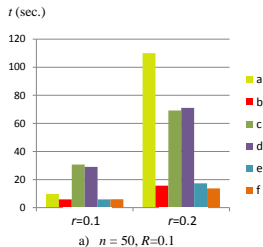
Example 2. Shary's interval system.



Basic algorithms:

- c** – interval Gauss method,
- d** – interval Gauss-Seidel method,
- e** – Hansen-Bleeck-Rohn procedure,
- f** – verifylss.

Example 3.



Basic algorithms:

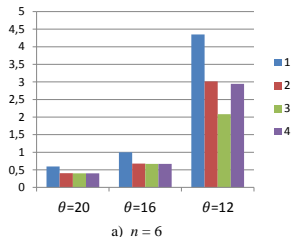
- a – Krawczyk method,
- b – modified Krawczyk method,
- c – interval Gauss method,
- d – interval Gauss-Seidel method,
- e – Hansen-Bleeck-Rohn procedure,
- f – verifyfls.

Influence of the basic methods

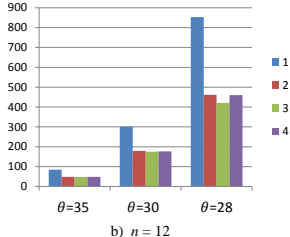
- The programs based on Krawczyk method and modified Krawczyk method are not effective for the matrices near the boundaries of singularity. For these matrices the programs based on interval Gauss and Gauss-Seidel methods work more quickly.
- For the matrices with near-zero ρ and sufficiently large $\Delta\sigma$, modified Krawczyk method is more preferable than interval Gauss and Gauss-Seidel methods. It works also better than Krawczyk method. However the use of these methods as the basic algorithms are not recommended.
- The procedure `verify1ss` from the MATLAB toolbox INTLAB in whole demonstrated quite good results.
- All numerical experiments show that **Hansen-Bleek-Rohn procedure** is the most effective basic method.

Influence of structure of the working list and processing

t (sec.)



t (sec.)



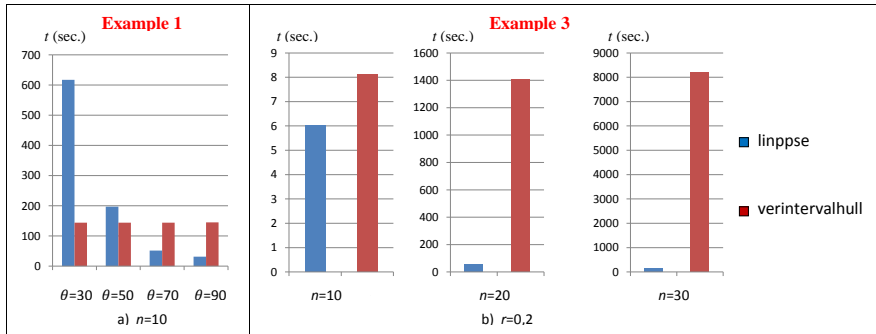
- The way of organising the list as a heap is the least efficient.
- The speed of list processing significantly increases if the records are ascending ordered with respect to the estimate $\Upsilon(Q, r)$.
- Pankov's way doesn't increase significantly the run time of the algorithm.
- The most preferable is the way in which the maximal length of the sublist of the active records is fixed.

Comparison of the algorithms for the optimal outer estimation of the solution set

We developed the algorithm `linppse` implemented the modification of PPS-method with Rohn's modification in which

- Hansen-Bleeck-Rohn procedure is used as the basic algorithm,
- in the working list the ordered sublist of the active records, which has the fixed maximal length, is separated, and the rest records are formed as a heap.

We compare it with the algorithm `verintervalhull` — procedure from the toolbox VERSOFT based on Rohn's method.



- The run time of the both algorithms grow exponentially with the dimension of the system.
- The run time of the procedure `verintervalhull` doesn't depend on the properties of the interval matrix \mathbf{A} and stay put for the different parameter values.
- The speed of the algorithm `linppse` slows down if the matrix \mathbf{A} is near the boundaries of singularity.
- For the large dimension n the procedure `linppse` is more efficient than `verintervalhull`.

Thank you

Computational complexity of optimal interval estimation

The problem of optimal interval estimation of the solution set to interval linear system is **NP-hard**, that is, it is an intractable problem, which requires times that are exponential functions of the problem size n^1 .

Numerical methods for solving these problems are similar to search algorithms of discrete optimization and a little more better than exhaustive search.

¹Kreinovich V., Lakeyev A.V., Rohn J., Kahl P. Computational complexity and feasibility of data processing and interval computations. – Dordrecht: Kluwer, 1997. 

Интервальные методы Гаусса и Гаусса-Зейделя

Интервальный метод Гаусса является интервальным аналогом хорошо известного в линейной алгебре метода исключения Гаусса, состоящего в преобразовании матрицы системы к верхнему треугольному виду (прямой ход) и последовательном вычислении значений неизвестных (обратный ход). Алгоритм метода такой же, как и в вещественном случае, только он оперирует интервальными величинами с помощью операций интервальной арифметики.

Интервальный метод Гаусса-Зейделя — итерационная процедура для уточнения внешней оценки множества решений.

Пусть $\mathbf{x}^{(k)} \supseteq \Xi(\mathbf{A}, \mathbf{b})$ и $\mathbf{A} = (a_{ij})$ такова, что $0 \notin a_{ii}$ для $i = 1, 2, \dots, n$. Уточненную оценку $\tilde{\mathbf{x}}$ находим следующим образом

$$\tilde{x}_i = x_i^{(k)} \cap \left(b_i - \sum_{j=1}^{i-1} a_{ij} \tilde{x}_j - \sum_{j=i+1}^n a_{ij} x_j^{(k)} \right) / a_{ii}, \quad i = 1, 2, \dots, n.$$

Если расстояние между векторами $\mathbf{x}^{(k)}$ и $\tilde{\mathbf{x}}$ больше заданной малой величины $\varepsilon > 0$, полагаем

$$\mathbf{x}^{(k+1)} \leftarrow \tilde{\mathbf{x}}, \quad k = 0, 1, 2, \dots$$

Метод Кравчика

Пусть интервальная линейная система $\mathbf{A}\mathbf{x} = \mathbf{b}$ предобусловлена обратной средней матрицей $C = (\text{mid } \mathbf{A})^{-1}$ и $\mathbf{x}^{(k)}$ — внешняя оценка множества решений $\Xi(\mathbf{A}, \mathbf{b})$.

Для уточнения оценки $\mathbf{x}^{(k)}$ применим **итерационный метод Кравчика**:

$$\mathbf{x}^{(k+1)} \leftarrow (C\mathbf{b} + (I - C\mathbf{A})\mathbf{x}^{(k)}) \cap \mathbf{x}^{(k)}, \quad k = 0, 1, 2, \dots,$$

где в качестве начального приближения можно взять вектор

$$\mathbf{x}^{(0)} = ([-\alpha, \alpha], \dots, [-\alpha, \alpha])^\top, \\ \alpha = \frac{\|C\mathbf{b}\|_\infty}{1 - \beta} \quad \text{и} \quad \beta = \|I - C\mathbf{A}\|_\infty < 1.$$

Итерационный процесс будет остановлен в том случае, когда расстояние между векторами $\mathbf{x}^{(k+1)}$ и $\mathbf{x}^{(k)}$ будет незначительно отличаться от нуля.

Модифицированный метод Кравчика

Пусть $\mathbf{d}^{(k)}$ — внешняя оценка множества решений $\Xi(\mathbf{A}, \mathbf{b} - \mathbf{A}x_s)$, где $x_s = C \cdot \text{mid } \mathbf{b}$.

Для уточнения оценки $\mathbf{d}^{(k)}$ применим **модифицированный метод Кравчика**:

$$\mathbf{d}^{(k+1)} \leftarrow (C(\mathbf{b} - \mathbf{A}x_s) + (I - CA)\mathbf{d}^{(k)}) \cap \mathbf{d}^{(k)}, \quad k = 0, 1, 2, \dots$$

Итерационный процесс осуществляется с помощью «эпсилон-раздутия», при котором интервал \mathbf{d} , полученный на текущем шаге алгоритма, заменяется на объемлющий интервал $\mathbf{d}_\varepsilon = \mathbf{d} + [-\varepsilon, \varepsilon] \text{rad } \mathbf{d} + [-\eta, \eta] e$, где ε, η — некоторые малые положительные вещественные числа, $e = (1, 1, \dots, 1)^\top$.

Полагая $\varepsilon = 0.1$ и η — наименьшее положительное вещественное число, представимое в ЭВМ, получим итерационную процедуру

$$\mathbf{d}^{(k+1)} \leftarrow C(\mathbf{b} - \mathbf{A}x_s) + (I - CA)(\mathbf{d}^{(k)} + 0.1[-1, 1] \text{rad } \mathbf{d} + [-10\eta, 10\eta] e),$$

$$k = 0, 1, 2, \dots,$$

критерием остановки которой служит условие: $\mathbf{d}^{(k+1)} \subseteq \mathbf{d}^{(k)}$.

Процедура Хансена-Блика-Рона

Процедура Хансена-Блика-Рона основана на следующем результате.

Теорема. Пусть матрица $\mathbf{A} = (\mathbf{a}_{ij}) \in \mathbb{IR}^{n \times n}$ является интервальной H -матрицей и

$$\begin{aligned}u_i &= (\langle \mathbf{A} \rangle^{-1} |\mathbf{b}|)_i, & d_i &= (\langle \mathbf{A} \rangle^{-1})_{ii}, \\ \alpha_i &= \langle \mathbf{a}_{ii} \rangle - 1/d_i, & \beta_i &= u_i/d_i - |\mathbf{b}_i|,\end{aligned}$$

где $\langle \mathbf{A} \rangle$ — компарант матрицы \mathbf{A} , $i = 1, 2, \dots, n$. Тогда множество решений $\Xi(\mathbf{A}, \mathbf{b})$ содержится в интервальном векторе $\mathbf{x} = (x_i)$ с компонентами

$$x_i = \frac{\mathbf{b}_i + \beta_i[-1, 1]}{\mathbf{a}_{ii} + \alpha_i[-1, 1]}, \quad i = 1, 2, \dots, n.$$

Для оценки сверху α_i и β_i вычисляется верхняя оценка $B = \tilde{B} + vv^T$ матрицы $\langle \mathbf{A} \rangle^{-1}$, где \tilde{B} — некоторая оценка $\langle \mathbf{A} \rangle^{-1}$, $w_k = \max_i \frac{-R_{ik}}{u_i}$, $R = \langle \mathbf{A} \rangle \tilde{B} - I$, $v = \tilde{B}u$, а в качестве u можно взять единичный вектор $(1, \dots, 1)$.

Comparison of the algorithms for the optimal outer estimation of the solution set

We compare the efficiency of the following algorithms:

linppsr — the procedure based on PPS-method with Rohn's modification, in which Hansen-Bleek-Rohn procedure is used as the basic algorithm and in the working list the ordered sublist of the active records, which has the fixed maximal length, is separated, and the rest records are formed as a heap;

verintervalhull — procedure from the toolbox VERSOFT based on Rohn's method.