Interval methods for computing various refinements of Nash equilibria

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Game theory

- Consider a situation when $n \geq 2$ (rational) agents interact with each other, i.e., when one agent's action depends essentially on what other agents may do.
- Game – a model of disagreement/conflict of interests between interacting agents.
- R.Aumann suggested to change "Game theory" for "Interactive decision theory".
- Antagonistic (strictly competitive, constant-sum) vs. non-antagonistic (non constant-sum) games.
- Cooperative vs. non-cooperative games.

The strategic (normal) model of the game

There are n agents, each of them equipped with

- a set of feasible alternatives X_i,
- the utility (for minimization – cost) function q_i

$$q_i : X_1 \times \ldots \times X_n \rightarrow \mathbb{R},$$

modeling the agent's preference.

- All agents are trying to choose their decision x_i in X_i to minimize their cost functions.

Solutions of a game

- The point (strategy assignment) that the players are going to choose (or we suppose them to do so), provided:
  - assumptions on their rationality,
  - assumptions on their knowledge,
  - ...
- Concepts:
  - Dominant strategy equilibrium.
  - The Nash equilibrium.
  - The core of a game (for cooperative games).
  - ...

Nash equilibrium

Let the game $(X_1,\ldots,X_n;q_1,\ldots,q_n)$ be given.

Decision $(x_1^*,\ldots,x_n^*)$ is a Nash equilibrium if
for all i and all x_i in X_i

$$q_i(x_1^*,\ldots,x_i,\ldots,x_n^*) \geq q_i(x_1^*,\ldots,x_i^*,\ldots,x_n^*).$$

We remember – the agent wants to minimize his cost function.

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When we denote

$$x_{\setminus i} = (x_{1,}\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

the following, more compact, restatement of the definition is useful.

Decision $(x_i^*, x_{\setminus i}^*)$ is a Nash equilibrium if for all i and all x_i in X_i:

$$q_i(x_i, x_{\setminus i}^*) \geq q_i(x_i^*, x_{\setminus i}^*).$$

Nash equilibrium

- To find Nash equilibrium, and especially all equilibria, for continuous games is a hard task.
- In the literature one can find a few approaches based on different theoretical frameworks:
  - minimization of function defined by Nikaidô and Isoda,
  - solving necessary differential optimality conditions (Fermat's theorem),
  - best replay dynamic (myopic behavior, индикаторное поведение),
  - our approach, using interval methods,
  - ...

Our interval algorithm

- Previous papers:
  - B. J. Kubica and A. Woźniak, "*An interval method for seeking the Nash equilibria of non-cooperative games*", PPAM 2009 Proceedings, LNCS, Vol. 6068, pp. 446 – 455 (2010).
  - B. J. Kubica and A. Woźniak, "*Applying an interval method for a four agent economy analysis*", PPAM 2011 Proceedings, LNCS, Vol. 7204, pp. 477 – 483 (2012).
- Idea: interval methods can be used to solve the following system of conditions:

$$\forall i=1,\ldots,n \;\; \forall x_i \in \boldsymbol{x_i} \subseteq \mathbb{R}^{k_i}$$

$$q_i\left(x_1^*,\ldots,x_{i-1}^*,x_i,x_{i+1}^*,\ldots,x_n^*\right) \geqslant q_i\left(x_1^*,\ldots,x_n^*\right)$$

Our interval algorithm

- The branch-and-bound (b&b) schema.
- Rejection/reduction tools:
  - An analog of the monotonicity test.
  - The Newton operator used to solve the system of necessary conditions for Nash points.
- A "second phase" to delete points that are not Nash equilibria, using 0th-order tools.
  - No simple approach.
  - A few possibilities (e.g., using an interval tree, to store cost values of different players for different parts of the domain, comparing only selected points, etc.).

Necessary conditions

- Well determined ($N = \sum_i k_i$ equations and total N variables).
- The Jacobi matrix is composed of rows of Jacobi matrices of systems $\nabla q_i(x_1, \ldots, x_n) = 0$

$$
\begin{array}{cccc}
\dfrac{\partial q_1(x)}{\partial x_1} = 0, & \dfrac{\partial q_1(x)}{\partial x_2} = 0, & \cdots & \dfrac{\partial q_1(x)}{\partial x_n} = 0, \\
\dfrac{\partial q_2(x)}{\partial x_1} = 0, & \dfrac{\partial q_2(x)}{\partial x_2} = 0, & \cdots & \dfrac{\partial q_2(x)}{\partial x_n} = 0, \\
\vdots & \vdots & \cdots & \vdots \\
\dfrac{\partial q_n(x)}{\partial x_1} = 0, & \dfrac{\partial q_n(x)}{\partial x_2} = 0, & \cdots & \dfrac{\partial q_n(x)}{\partial x_n} = 0.
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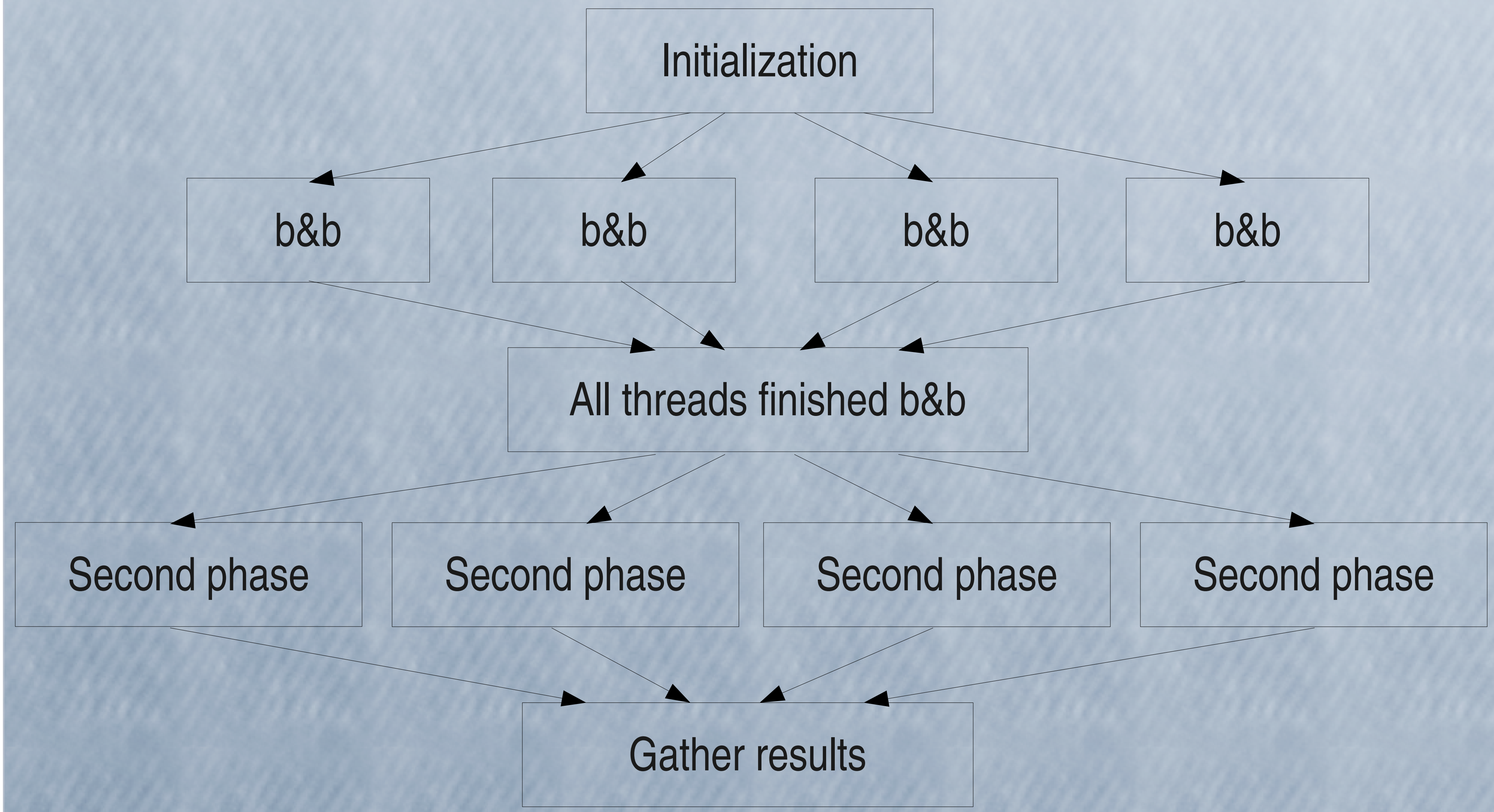
\end{array}
$$

The overall algorithm

```
L = {}; Lsol = {};
// branch-and-bound
enqueue (L, x);
while nonempty(L) do
        dequeue (L, x);
        narrow (x); // using monot. test and Nash-point necessary conditions
        if (x was discarded) then cycle;
        if (diam (x) < ε) then enqueue (Lsol, x);
        else
                bisect (x, x1, x2);   enqueue (L, x2);   x = x1;
        end
end while
// Second phase
foreach x in Lsol
        if (x cannot contain a Nash equilibrium) then discard x;
end foreach
```

Parallelization



The concavity test

- Used for global optimization.
- Also known as "non-convexity test"; both names are confusing...
- Essence: check if the objective function cannot be convex at any point of the given box (i.e., it is concave with respect to at least one of the variables).
- A simple version – check if there is a strictly negative element on main diagonal of the Hesse matrix.
- Alternatively, we can compute eigenvalues, but that is time consuming (and usually inefficient).

The concavity test

- For global optimization the concavity test is not very useful (at least, according to our experiments).
  - It deletes few boxes.
  - The midpoint test (a 0^{th} order tool) would also delete these boxes, simply.
- But for the Nash equilibria seeking problem, we do not have efficient 0^{th} order tools!
  - Hence, the analog of concavity test becomes important.
- For seeking plain Nash points, we consider concavity of players' costs wrt their control variables.
- For the refinements, the procedure might be a bit different...

Drawbacks of the notion of Nash equilibrium

- There can be no Nash equilibrium.
- Also, there can exist many Nash equilibria.
  - We do not know, which of them the players will choose, actually.
- A Nash equilibrium does not have to be Pareto-optimal.
  - So, it might be a quite undesirable point.
- Nash equilibrium concept assumes perfect rationality of all players and finding it requires complete information, often.
- ...

Variants of the Nash equilibrium

- Epsilon-equilibrium – $q_i\left(x_i, x^*_{\setminus i}\right) \geq q_i\left(x^*_i, x^*_{\setminus i}\right) - \varepsilon$.
  - Some games (at least stochastic ones) that do not have Nash equilibria have an epsilon-equilibrium, e.g., the matching pennies game.
- Strong Nash equilibrium (SNE):
  - A Nash equilibrium fulfilling some additional requirement:

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- Strong Nash equilibrium (SNE):
  - A Nash equilibrium fulfilling some additional requirement:

Not only for a single player it is not beneficial to change their SNE strategy, but also for each member of every conceivable coalition S of players

$$\forall S \ \ \forall i \in S \ \ \forall x_S \ \ q_i(x_S, x^*_{\setminus S}) \geq q_i(x^*_S, x^*_{\setminus S}).$$

Inventor of the SNE

In many real-world situations, cooperation may be easier to sustain in a long-term relationship than in a single encounter. Analyzes of short-run games are, thus, often too restrictive.
Robert Aumann was the first to conduct a full-fledged formal analysis of so-called infinitely repeated games. His research identified exactly what outcomes can be upheld over time in long-run relations.

From: Prize in Memory of Alfred Nobel Announcement, 10 October 2005.

Robert John (Yisrael) Aumann
(born 1930)

Strong Nash equilibriua – comments

- The notion of SNE is very "strong", indeed – SNEs have to be Pareto-optimal.
- Actually, the notion is "too strong" – SNEs rarely exist!
- Yet there are some games, for which strong Nash equilibria are guaranteed to exist, e.g., some population games.
- An obvious weakening: *k*-equilibrium (*k*-SNE): we consider only coalitions of at most *k* players.

Computing strong Nash equilibria

- Necessary conditions:
  - All conditions for ordinary Nash equilibria hold!
  - And there are additional ones.
- So, the system is overdetermined.
  - That is the reason (at least one of them) why SNEs exist so rarely.
  - It will not be possible to compute verified results using the interval Newton operator.
- What are these necessary conditions, specifically?

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  - It will not be possible to compute verified results using the interval Newton operator.
- What are these necessary conditions, specifically?
- We assume *i*-th player controls the variable x_i; extension to the general case is straightforward.

Necessary conditions for a 2-SNE

$$\frac{\partial q_1(x)}{\partial x_1}=0, \quad \frac{\partial q_2(x)}{\partial x_2}=0, \quad \cdots, \quad \frac{\partial q_n(x)}{\partial x_n}=0,$$

For each pair (i, j) of players (q_i, q_j) is Pareto-optimal with respect to (x_i, x_j), which can be expressed by the necessary conditions for Pareto-optimality: $\forall i, j \quad i \neq j$

$$u_i^{(ij)} \cdot \frac{\partial q_i(x)}{\partial x_i} + u_j^{(ij)} \cdot \frac{\partial q_j(x)}{\partial x_i} = 0,$$

$$u_i^{(ij)} \cdot \frac{\partial q_i(x)}{\partial x_j} + u_j^{(ij)} \cdot \frac{\partial q_j(x)}{\partial x_j} = 0,$$

$$u_i^{(ij)} + u_j^{(ij)} = 1.$$

Necessary conditions for a 2-SNE

Which results in:

$$u_j^{(ij)} \cdot \frac{\partial q_j(x)}{\partial x_i} = 0,$$

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$$u_i^{(ij)} + u_j^{(ij)} = 1.$$

And, as $u_i^{(ij)}$ and $u_j^{(ij)}$ cannot be both equal to zero, we obtain the condition:

$$\frac{\partial q_j(x)}{\partial x_i} = 0 \text{ or } \frac{\partial q_i(x)}{\partial x_j} = 0 \quad \forall i, j \quad i \neq j.$$

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Interpretation: for no pair of players it is possible that they mutually reduce each other's cost.

Solving necessary conditions for a 2-SNE

- We can solve the condition $\left(\frac{\partial q_j(x)}{\partial x_i}\right) \cdot \left(\frac{\partial q_i(x)}{\partial x_j}\right) = 0$.
- This would be however less efficient (we loose some important information).
- A new monotonicity test (check if zero does not belong to at least one of the gradients' components of the pair).
- The Newton operator for an alternative of equations!
  - More possible results than for a traditional Newton operator (the box can be contracted and split at the same time!).

Necessary conditions for a *k*-SNE

- For $k > 2$ we get yet more necessary conditions ad they are yet more complicated.
- A "full" SNE has to fulfill them all!
- 0^{th} order procedures are yet more complicated – none of the pairs of cost functions can be dominated!
- Solving these conditions (or their form) will not be presented here – the work is in progress.

A simple example

- Player 1 controls variable x_1, player 2 – x_2.
- Both objectives are minimized:

$$q_1(x_1, x_2) = x_1^4 - 3.75\, x_1^3 + 3.25\, x_1^2 + x_2^2 + 1,$$
$$q_2(x_1, x_2) = x_2^4 - 3.75\, x_2^3 + 3.25\, x_2^2 + x_1^2 + 1,$$
$$x_1, x_2 \in [-3, 3].$$

- The game has a single Nash equilibrium (2, 2), but it is not a strong Nash equilibrium.
- The point (0, 0) fulfills the necessary condition for a SNE, but it is not a Nash point (players benefit from deviating their control from 0 to 2).

A simple example

- The algorithm for Nash points finds three points, quickly: (2, 2), (0, 0) and (0, 2).
- The proposed "monotonicity test" for strong Nash points discards the point (2, 2), efficiently. Other points can be discarded by comparing the values of functions $q_1(.)$, $q_2(.)$.
- Obviously, for some problems the algorithm may be inefficient, but this requires further research.

Suggestion the notion of epsilon-SNE (epsilon-*k*-SNE)

- We cannot solve an overdetermined system precisely, but we can solve it approximately – and it is quite easy!
  - Minimise a norm (e.g., quadratic or Chebyshev) of the vector of all functions.
- Such equilibira may exist more often; consequently – be more useful than classical SNE.
- Not investigated yet?

Further research

- Considering more sophisticated examples of strong Nash equilibria computations and tuning our algorithm for these cases.
  - In particular, investigating the interval Newton operator applied to an alternative of equations.
- Investigating the notion of epsilon-SNE:
  - Its theoretical properties.
  - Possibilities of finding it numerically.

Conclusions

- Interval methods are well suited to seek points that fulfill a certain condition – in particular Nash equilibria of a game and its various modifications.
- Here, we proposed an interval algorithm for seeking strong Nash equilibria of a continuos game.
- The tools to be applied and some preliminary results have been presented.
- Some insight on the theory (the notion of epsilon-SNE) was considered, also.