Reduction of the Pareto Set in Multicriteria Economic Problem with CES Functions

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Production functions

- Cobb and Douglas
- Leontief
- CES function
- Mukerji
- non-homotetic CES function
- translogarithmic function
- VES function
- generalized Leontief function

$$\text{PROFIT} = \text{INCOME} - \text{COSTS} \longrightarrow \max$$

Multicriteria model $< X, f >$

X is a set of feasible solutions (alternatives), $X \subseteq \mathbb{R}^n$;
$f = (f_1, \ldots, f_m)$ is a vector criterion, $f \colon X \to \mathbb{R}^m$.

The Pareto set (in terms of X)

$P_f(X) = \{x \in X \mid \nexists x^* \in X : f(x^*) \geq f(x)\}$.

A set of feasible outcomes $Y = f(X)$.

The Pareto set (in terms of Y)

$P(Y) = \{y \in Y \mid \nexists y^* \in Y : y^* \ \geq \ y \ \}$.

$y^* \geq y$: $\quad y_i^* \geqslant y_i$ for all $i \in \{1, 2, \ldots, m\}$, and $y^* \neq y$.

Multicriteria choice model $< X, f, \succ >$

- a set of feasible solutions (alternatives) X;
- a vector criterion $f = (f_1, f_2, \ldots, f_m)$ defined on set X;
- an asymmetric binary preference relation of the DM $\succ$ defined on set $Y = f(X)$,

Preference relation $\succ$ of the DM [Noghin]

$f(x') \succ f(x'')$: x' is preferred to x''
A1: $f(x') \succ f(x'') \;\Rightarrow\; x'' \notin$ opt. choice within the whole set X.
A2: $\succ$ is irreflexive and transitive.
A3: $\succ$ is compatible with each criteria $f_1, f_2, \ldots, f_m$.
A4: $\succ$ is invariant under linear positive transformation.

Reduction of the Pareto set

The Edgeworth–Pareto principle [Noghin]

$$C(X) \subseteq P_f(X) \quad \forall x \in X, \forall C(\cdot) \text{ under } \mathbf{A1–A4},$$

where $C(X)$ is an optimal choice.

"Quantum of information" about relation $\succ$ [Noghin]

$$A \longrightarrow B \text{ with } w_i,\ w_j > 0$$

$$y \in \mathbb{R}^m : \quad y_i = w_i, \quad y_j = -w_j, \quad y_s = 0 : \quad y \succ 0;$$

$$\forall i \in A \quad \forall j \in B \quad \forall s \notin A, B;$$

where $I = \{1, 2, \dots, m\}$, $A, B \subset I$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$.

"New" multicriteria problem $< X, g >$

$$g = F\left(f(\cdot), \{w_i \ \forall i \in A\}, \{w_j \ \forall j \in B\}\right)$$

$$C(X) \subseteq P_g(X) \subseteq P_f(X)$$

Statement of economic model

Denote by K and L quantities of the basic production assets and the labor resources respectively.

Production CES function

$$Q = F(aK^{-r} + (1-a)L^{-r})^{-(1/r)},$$

where Q is a quantity of output, F is a factor productivity, a is a share parameter, r is a parameter such that the quantity $1/(1+r)$ is the elasticity of substitution.
$F > 0,\ 0 < a < 1,\ r > -1.$

Objective functions (criteria)

$f \;=\; (f_1, f_2, f_3);$
f_1 is the labor costs, f_2 is the costs of the basic production assets, f_3 is the cost of the manufactured products.
$f_1(K, L) = -p_{_K} K, \quad f_2(K, L) = -p_{_L} L,$
$f_3(K, L) = p_{_Q} Q = p_{_Q} F(aK^{-r} + (1-a)L^{-r})^{(-1/r)}.$

Multicriteria economic choice problem

- $X = \{x = (K, L) \in \mathbb{R}^2 \mid K, L > 0\}$;
- $f(K, L) = (f_1(K, L), f_2(K, L), f_3(K, L))$ defined on set X;
- preference relation of the DM $\succ$ defined on set $Y = f(X)$,

Two "quanta of information" about relation $\succ$

$$(P1): \quad y^{(1)} = \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \\ -w_3^{(1)} \end{pmatrix}, \quad (P2): \quad y^{(2)} = \begin{pmatrix} -w_1^{(2)} \\ -w_2^{(2)} \\ w_3^{(2)} \end{pmatrix},$$

such that the relations $y^{(1)} \succ 0_3$ and $y^{(2)} \succ 0_3$ hold.

Condition of consistency

$$w_1^{(1)}/w_3^{(1)} > w_1^{(2)}/w_3^{(2)}, \quad w_2^{(1)}/w_3^{(1)} > w_2^{(2)}/w_3^{(2)}.$$

Using (P1), (P2) information

Note, that $P_f(X) = X$.

Theorem [Klimova, Noghin]

Given information (P1) and (P2), the condition of consistency holds.
$C(X) \subseteq P_g(X) \subseteq P_f(X)$, where $g = (g_{13}, g_{23}, g_{31}, g_{32})$:

$$g_{13} = w_1^{(1)} f_3 + w_3^{(1)} f_1, \quad g_{23} = w_2^{(1)} f_3 + w_3^{(1)} f_2,$$
$$g_{31} = w_1^{(2)} f_3 + w_3^{(2)} f_1, \quad g_{32} = w_2^{(2)} f_3 + w_3^{(2)} f_2.$$

$$g_{13}(K, L) = w_1^{(1)} p_Q F(aK^{-r} + (1-a)L^{-r})^{\frac{1}{-r}} - w_3^{(1)} p_K K,$$
$$g_{23}(K, L) = w_2^{(1)} p_Q F(aK^{-r} + (1-a)L^{-r})^{\frac{1}{-r}} - w_3^{(1)} p_L L,$$
$$g_{31}(K, L) = w_1^{(2)} p_Q F(aK^{-r} + (1-a)L^{-r})^{\frac{1}{-r}} - w_3^{(2)} p_K K,$$
$$g_{32}(K, L) = w_2^{(2)} p_Q F(aK^{-r} + (1-a)L^{-r})^{\frac{1}{-r}} - w_3^{(2)} p_L L.$$

Finding the Pareto set $P_g(X)$

The set of feasible solutions X is convex; functions $g_{13}(K, L)$, $g_{23}(K, L)$, $g_{31}(K, L)$, and $g_{32}(K, L)$ are concave on set X.
$\Rightarrow$ Theorem of Karlin and Hurwicz:

$$P_g(X) = \arg\max_{x \in X}(\varphi(x)),$$

where
$\varphi(x) = \lambda_1 g_{13}(K, L) + \lambda_2 g_{23}(K, L) + \lambda_3 g_{31}(K, L) + \lambda_4 g_{32}(K, L)$,
$\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$, $\Sigma_{i=1}^4 \lambda_i = 1$.

$$L_0 = K_0 \left(\left(\frac{(1-a)p_K(\lambda_1 w_3^{(1)} + \lambda_3 w_3^{(2)})}{a p_L(\lambda_2 w_3^{(1)} + \lambda_4 w_3^{(2)})} \right)^{\frac{-r}{1+r}} - \frac{a}{1-a} \right)^{\frac{-1}{r}} \quad (1)$$

$$P_g(X) = \{(K_0, L_0) \in X \mid (1) \text{ holds}\}$$

Fuzzy multicriteria choice model $< X, f, \mu >$

- a set of feasible solutions (alternatives) X;
- a vector criterion $f = (f_1, f_2, \ldots, f_m)$ defined on set X;
- a fuzzy preference relation of the DM μ defined on set $Y = f(X)$.

Fuzzy preference relation μ of the DM [Noghin]

$$\mu \colon Y \times Y \to [0, 1]$$

$\mu(f(x'), f(x'')) = \mu^* : x'$ is preferred to x'' with degree of confidence μ^*.

A1: $\mu(f(x'), f(x'')) = \mu^* \;\Rightarrow\; x'' \notin$ opt. choice within the whole set X with degree of confidence $1 - \mu^*$.

A2: μ is irreflexive and transitive.

A3: μ is compatible with each criteria $f_1, f_2, \ldots, f_m$.

A4: μ is invariant under linear positive transformation.

Reduction of the Pareto set (fuzzy case)

The Edgeworth–Pareto principle [Noghin]

$$\lambda_X^C(x) \leqslant \lambda_X^P(x) \quad \forall x \in X, \quad \forall \lambda_X^C(\cdot) \text{ under } \mathbf{A1–A4},$$

where $\lambda_X^C(\cdot)$ is a membership function of optimal choice.

"Fuzzy quantum of information" about relation μ [Noghin]

$$A \longrightarrow B \text{ with } w_i,\, w_j > 0,\ \mu^* \in [0,1]$$
$$y \in \mathbb{R}^m : \quad y_i = w_i, \quad y_j = -w_j, \quad y_s = 0 : \quad \mu(y, 0_m) = \mu^*;$$
$$\forall i \in A \quad \forall j \in B \quad \forall s \notin A, B;$$

where $I = \{1, 2, \ldots, m\}$, $A, B \subset I$, $A \neq \emptyset$, $B \neq \emptyset$, $A \cap B = \emptyset$.

Series of "new" multicriteria problems $< X, g^l >$

$$g^l = F^l\left(f(\cdot), \{w_i\ \forall i \in A\}, \{w_j\ \forall j \in B\}\right), \ l \in \{1, 2, \ldots, k\}$$

$$C(X) \subseteq P_{g^1}(X) \subseteq P_{g^2}(X) \subseteq \ldots \subseteq P_{g^k}(X) \subseteq P_f(X)$$

Fuzzy multicriteria economic choice problem

- $X = \{x = (K, L) \in \mathbb{R}^2 \mid K, L > 0\}$;
- $f(K, L) = (f_1(K, L), f_2(K, L), f_3(K, L))$ defined on set X;
- fuzzy preference relation of the DM μ defined on set $Y = f(X)$,

Two "fuzzy quanta of information" about relation μ

$$(FP1): \quad y^{(1)} = \begin{pmatrix} w_1^{(1)} \\ w_2^{(1)} \\ -w_3^{(1)} \end{pmatrix}, \quad (FP2): \quad y^{(2)} = \begin{pmatrix} -w_1^{(2)} \\ -w_2^{(2)} \\ w_3^{(2)} \end{pmatrix},$$

such that the relations $\mu(y^{(1)}, 0_3) = \mu_1$ and $\mu(y^{(2)}, 0_3) = \mu_2$ hold.

Condition of consistency

$$w_1^{(1)}/w_3^{(1)} > w_1^{(2)}/w_3^{(2)}, \quad w_2^{(1)}/w_3^{(1)} > w_2^{(2)}/w_3^{(2)}.$$

Using (FP1), (FP2) information ($\mu_1 \geqslant \mu_2$)

Construct $\lambda_X^M(\cdot)$ of fuzzy set M.

1. Problem $< X, f, \succ >$

No additional information.
$P_f(X) = X \quad \lambda_X^M(x) = 0 \quad \forall x \in X \setminus P_f(X).$

2. Problem $< X, \bar{f}, \succ >$

$< X, f, \succ > \quad + \quad$ (P1)
$\bar{f} = (\bar{f}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4) : \bar{f}_1 = f_1,\ \bar{f}_2 = f_2,\ \bar{f}_3 = g_{13} = w_1^{(1)} f_3 + w_3^{(1)} f_1,$
$\bar{f}_4 = g_{23} = w_2^{(1)} f_3 + w_3^{(1)} f_2.$
$\lambda_X^M(x) = 1 - \mu_1 \quad \forall x \in P_f(X) \setminus P_{\bar{f}}(X)$

3. Problem $< X, g, \succ >$

$< X, f, \succ > \quad + \quad$ (P1) and (P2)
$\lambda_X^M(x) = 1 - \mu_2 \quad \forall x \in P_{\bar{f}}(X) \setminus P_g(X),\ \lambda_X^M(x) = 1 \quad \forall x \in P_g(X).$

Using (FP1), (FP2) information ($\mu_1 \geqslant \mu_2$)

We get the collection of nested sets:

$$C(X) \subseteq P_g(X) \subseteq P_{\bar{f}}(X) \subseteq P_f(X),$$

or in terms of membership functions

$$\lambda_X^C(x) \leqslant \lambda_X^M(x) \leqslant \lambda_X^P(x) \quad \forall x \in X.$$

$\lambda_X^M(\cdot)$ is a new fuzzy upper bound of the optimal choice

Constructing membership function $\lambda_X^M(\cdot)$

$\lambda_X^M(x) = 1 - \mu_1 \quad \forall x = (K, L):$

$L = K\left(\left(\frac{(1-a)p_K\lambda_{01}}{ap_L\lambda_{02}}\right)^{\frac{-r}{1+r}} - \frac{a}{1-a}\right)^{\frac{-1}{r}}$ for all positive λ_{01}, λ_{02},
no such positive quantities λ_1, λ_2, λ_3, and λ_4 that
$(\lambda_1 + w_3^{(1)}\lambda_2)\lambda_{01} = = (\lambda_2 + \lambda_4 w_3^{(1)})\lambda_{02}$

$\lambda_X^M(x) = 1 - \mu_2 \quad \forall x = (K, L):$

$$L = K\left(\left(\frac{(1-a)p_K(\lambda_2+\lambda_4 w_3^{(1)})}{ap_L(\lambda_1+\lambda_3 w_3^{(1)})}\right)^{\frac{-r}{1+r}} - \frac{a}{1-a}\right)^{\frac{-1}{r}}$$

for all positive λ_1, λ_2, λ_3, and λ_4, $\Sigma_{i=1}^4 \lambda_i = 1$,
no such positive quantities λ_{13}, λ_{23}, λ_{31}, and λ_{32} that

$$(\lambda_1 + \lambda_3 w_3^{(2)})(\lambda_{13} w_3^{(1)} + \lambda_{31} w_3^{(2)}) =$$
$$= (\lambda_2 + \lambda_4 w_3^{(1)})(\lambda_{23} w_3^{(1)} + \lambda_{32} w_3^{(2)}).$$

Constructing membership function $\lambda_X^M(\cdot)$

$\lambda_X^M(x) = 1 \quad \forall x = (K, L):$

$$L = K\left(\left(\frac{(1-a)p_K(\lambda_{13}w_3^{(1)}+\lambda_{31}w_3^{(2)})}{ap_L(\lambda_{23}w_3^{(1)}+\lambda_{32}w_3^{(2)})}\right)^{\frac{-r}{1+r}} - \frac{a}{1-a}\right)^{\frac{-1}{r}}$$

for all positive λ_{13}, λ_{23}, λ_{31}, and λ_{32},
$\lambda_{13} + \lambda_{23} + \lambda_{31} + \lambda_{32} = 1.$

Using (FP1), (FP2) information ($\mu_1 < \mu_2$)

Construct $\lambda_X^M(\cdot)$ of fuzzy set M.

1. Problem $< X, f, \succ >$

No additional information.
$P_f(X) = X \quad \lambda_X^M(x) = 0 \quad \forall x \in X \setminus P_f(X).$

2. Problem $< X, \hat{f}, \succ >$

$< X, f, \succ > \quad + \quad$ (P2)
$\hat{f} = (\hat{f}_1, \hat{f}_2, \hat{f}_3) : \hat{f}_1 = g_{31} = w_1^{(2)} f_3 + w_3^{(2)} f_1,$
$\hat{f}_2 = g_{32} = w_2^{(2)} f_3 + w_3^{(2)} f_2, \; \hat{f}_3 = f_3$
$\lambda_X^M(x) = 1 - \mu_2 \quad \forall x \in P_f(X) \setminus P_{\hat{f}}(X)$

3. Problem $< X, g, \succ >$

$< X, f, \succ > \quad + \quad$ (P1) and (P2)
$\lambda_X^M(x) = 1 - \mu_1 \quad \forall x \in P_{\hat{f}}(X) \setminus P_g(X), \; \lambda_X^M(x) = 1 \quad \forall x \in P_g(X).$

Conclusions

1. We consider 3-criteria economic choice problem with CES production function
2. We introduce the DM's preference model, where the group of the resources costs is more important then the income with corresponding parameters, and vice a versa
3. We investigate the reduction of the Pareto set using aforementioned model of preferences and evaluate it analytically
4. We consider the fuzzy preference model generalizing aforementioned information
5. We show how use the fuzzy preference information and construct analytically the membership function of fuzzy set, which is an bound for the optimal choice

Thank you for your attention!