Statistical Modeling of Random Processes with Invariants

T. A. Averina,
ICM & MG SB RAS, Novosibirsk State University, Novosibirsk
E. V. Karachanskaya,
Far Eastern State Transport University, Khabarovsk
K. A. Rybakov,
Moscow Aviation Institute, Moscow

20 September 2017

Stochastic differential equations with a given first integral

Dynamical system is described by the Itô SDE:

$$dX(t) = f\big(t, X(t)\big)dt + \sigma\big(t, X(t)\big)dW(t), \quad X(t_0) = X_0, \quad (1)$$

where

- $X \in \mathbb{R}^n$ is a state,
- $t \in [t_0, T]$ is a time,
- $f(t, x)\colon [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is an n-dimensional function,
- $\sigma(t, x)\colon [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times s}$ is an $(n \times s)$-dimensional matrix function,
- $W(t)$ is an s-dimensional Wiener process,
- $X_0 \in \mathbb{R}^n$ is an initial state ($W(t)$ and X_0 are independent).

Stochastic differential equations with a given first integral

Corresponding Stratonovich SDE:

$$dX(t) = a\big(t, X(t)\big)dt + \sigma\big(t, X(t)\big) \circ dW(t), \qquad (2)$$

where $a(t,x)\colon [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is an n-dimensional function.

Functions $f(t,x)$ and $a(t,x)$ satisfy the relation:

$$a_i(t,x) = f_i(t,x) - \frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{s}\frac{\partial\sigma_{il}(t,x)}{\partial x_j}\sigma_{jl}(t,x), \quad i = 1, 2, \ldots, n.$$

Stochastic differential equations with a given first integral

According to V. A. Dubko, a nonrandom function $M(t, x)$ is called **a first integral**[1,2,3] for the stochastic dynamical system (1) if $M(t, x) \neq \text{const}$ and this function equals a constant depending only on X_0 for all paths of the random process $X(t)$:

$$M\big(t, X(t)\big) = M(t_0, X_0) \quad \text{with probability 1.} \tag{3}$$

---

[1] V. A. Dubko, "Integral invariants for one class of systems of stochastic differential equations," *Dopov. Nats. Akad. Nauk Ukr. Mat. Tekh. Nauki*, no. 1, pp. 17–20, 1984.

[2] N. V. Krylov, B. L. Rozovskii, "Stochastic partial differential equations and diffusion processes," *Russian Math. Surveys*, vol. 37, no. 6, pp. 81–105, 1982.

[3] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, 1981.

Stochastic differential equations with a given first integral

Let $M(t,x)\colon [t_0,T]\times\mathbb{R}^n \to \mathbb{R}$ be a scalar nonrandom function, $M(t,x)\in C^{1,2}([t_0,T]\times\mathbb{R}^n)$, $M(t,x)$ is the first integral for the stochastic dynamical system (1). Therefore, $dM(t,X(t))=0$, and

$$\sum_{i=1}^{n}\sigma_{il}(t,x)\frac{\partial M(t,x)}{\partial x_i}=0, \quad l=1,2,\ldots,s; \tag{4}$$

$$\frac{\partial M(t,x)}{\partial t}+\sum_{i=1}^{n}\Big[f_i(t,x)-\frac{1}{2}\sum_{j=1}^{n}\sum_{l=1}^{s}\frac{\partial\sigma_{il}(t,x)}{\partial x_j}\sigma_{jl}(t,x)\Big] \times\frac{\partial M(t,x)}{\partial x_i}=0. \tag{5}$$

Stochastic differential equations with a given first integral

If $M(t,x)$ is the first integral for the stochastic dynamical system (2), then the condition

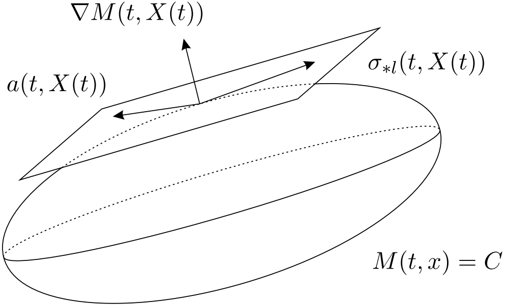
$$\frac{\partial M(t,x)}{\partial t} + \sum_{i=1}^{n} a_i(t,x)\frac{\partial M(t,x)}{\partial x_i} = 0 \qquad (6)$$

can be used instead of (5).

$\nabla M(t,X(t)) \perp a(t,X(t))$

$\nabla M(t,X(t)) \perp \sigma_{*l}(t,X(t))$

$l = 1, 2, \ldots, s$



Two-dimensional dynamical system

Dynamical system is defined by the following Itô SDEs ($n = 2$):

$$dX_i(t) = f_i\big(t, X_1(t), X_2(t)\big)dt + \sigma_i\big(t, X_1(t), X_2(t)\big)dW(t), \quad X_i(0) = X_{i0}, \quad (7)$$

where $W(t)$ is a scalar Wiener process ($s = 1$), $i = 1, 2$.
Corresponding Stratonovich SDEs:

$$dX_i(t) = a_i\big(t, X_1(t), X_2(t)\big)dt + \sigma_i\big(t, X_1(t), X_2(t)\big) \circ dW(t), \quad X_i(0) = X_{i0}, \quad (8)$$

i.e., $X(t) = [X_1(t)\ X_2(t)]^{\mathrm{T}}$, $X_0 = [X_{10}\ X_{20}]^{\mathrm{T}}$, and

$$f(t, x) = [f_1(t, x_1, x_2)\ f_2(t, x_1, x_2)]^{\mathrm{T}},$$
$$a(t, x) = [a_1(t, x_1, x_2)\ a_2(t, x_1, x_2)]^{\mathrm{T}},$$
$$\sigma(t, x) = [\sigma_1(t, x_1, x_2)\ \sigma_2(t, x_1, x_2)]^{\mathrm{T}}.$$

Two-dimensional dynamical systems

All functions in the equations (7) and (8) are defined by the special algorithm[4,5]. If $n = 2$ and $s = 1$, then

$$a_1(t, x_1, x_2) = \frac{H_1(t, x_1, x_2)}{C(t, x_1, x_2)}, \quad a_2(t, x_1, x_2) = \frac{H_2(t, x_1, x_2)}{C(t, x_1, x_2)},$$

where

$$H_1(t, x_1, x_2) = q_1(t, x_1, x_2)\frac{\partial M(t, x_1, x_2)}{\partial x_2}, \quad H_2(t, x_1, x_2) = -q_1(t, x_1, x_2)\frac{\partial M(t, x_1, x_2)}{\partial x_1},$$

$$C(t, x_1, x_2) = q_3(t, x_1, x_2)\frac{\partial M(t, x_1, x_2)}{\partial x_1} - q_2(t, x_1, x_2)\frac{\partial M(t, x_1, x_2)}{\partial x_2},$$

and

$$\sigma_1(t, x_1, x_2) = q_0(t, x_1, x_2)\frac{\partial M(t, x_1, x_2)}{\partial x_2}, \quad \sigma_2(t, x_1, x_2) = -q_0(t, x_1, x_2)\frac{\partial M(t, x_1, x_2)}{\partial x_1}.$$

[4] E. V. Chalykh, "Constructing the set of program controls with probability 1 for one class of stochastic systems," *Autom. Remote Control*, vol. 70, no. 8, pp. 1364–1375, 2009.

[5] E. V. Karachanskaya, *Integral Invariants of Stochastic Systems and Program Control with Probability 1*, Pacific National University, 2015.

Methodology of numerical experiments

Let $\{t_k\}$ be a discretization of the time interval $[t_0, T]$ with a step size h:

$$t_{k+1} = t_k + h, \quad k = 0, 1, \ldots, N-1, \quad t_N = T, \quad N = \frac{T - t_0}{h}.$$

Denote by $\{X_k\}$ a discrete-time approximation for the random process $X(t)$ determined by a numerical method for SDEs (7) or (8), i.e., the random vector X_k is an approximation of $X(t)$ at time t_k.
The discrete-time approximation $\{X_k\}$ converges with strong order p to the solution $X(t)$ at time $t = t_N = T$, if there exists a constant $c > 0$, such that $\varepsilon = \mathbb{E}\big[|X(T) - X_N|\big] \leqslant ch^p$.
Further, an another definition for the order of converges is used:

$$\begin{aligned}\varepsilon_{\mathcal{M}} &= \mathbb{E}\big[|M\big(T, X(T)\big) - M(T, X_N)|\big] \\ &= \mathbb{E}\big[|M(t_0, X_0) - M(T, X_N)|\big] \leqslant c^* h^p, \quad c^* > 0.\end{aligned}$$

Methodology of numerical experiments

1. Numerical methods for Itô SDEs:
   - Euler–Maruyama method,
   - Milstein method,
   - Platen method,
   - Artemiev method.

2. Numerical methods for Stratonovich SDEs:
   - Heun method,
   - Derivative-free Heun method,
   - Artemiev method,
   - Averina method.

Numerical method

Rosenbrock type method[6]:

$$X_{k+1} = X_k + \frac{h}{2}\left[I - \frac{h}{2}\frac{\partial a(t_k, X_k)}{\partial x}\right]^{-1}\left[a(t_k, X_k) + a(t_k, X_k^p)\right] + \frac{\sqrt{h}}{2}\left(\sigma(t_k, X_k) + \sigma(t_k, X_k^p)\right)\Delta W_k,$$

$$X_k^p = X_k + \sqrt{h}\sigma(t_k, X_k)\Delta W_k,$$

where h is a time step size, ΔW_k is a random value with a standard normal distribution ($s = 1$), and I is the (2×2)-dimensional identity matrix.

[6] T. A. Averina, *Construction of Statistical Modeling Algorithms for Systems with Random Structure*, Novosibirsk State University, 2015.

Examples: Elliptic cylinder

Manifold equation (invariant): $M(t, x_1, x_2) = x_1^2 + x_2^2 = C = \text{const}.$

The corresponding drift and diffusion coefficients are

$$f_1(t, x_1, x_2) = \frac{x_2 q_1(t, x_1, x_2)}{x_1 q_3(t, x_1, x_2) - x_2 q_2(t, x_1, x_2)} - 2x_1 q_0^2(t, x_1, x_2),$$

$$f_2(t, x_1, x_2) = -\frac{x_1 q_1(t, x_1, x_2)}{x_1 q_3(t, x_1, x_2) - x_2 q_2(t, x_1, x_2)} - 2x_2 q_0^2(t, x_1, x_2),$$

$$\sigma_1(t, x_1, x_2) = 2x_2 q_0(t, x_1, x_2), \quad \sigma_2(t, x_1, x_2) = -2x_1 q_0(t, x_1, x_2),$$

where $q_0(t, x_1, x_2) = 1/\sqrt{2}$, $q_1(t, x_1, x_2) = 1$, $q_2(t, x_1, x_2) = 1/x_2$, and $q_3(t, x_1, x_2) = 2/x_1$.

Itô and Stratonovich SDEs:

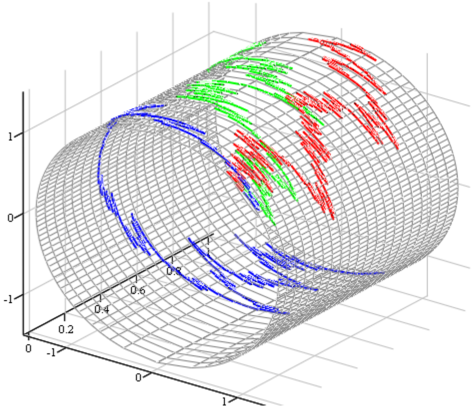
$$dX_1(t) = \big(-X_1(t) + X_2(t)\big)dt + \sqrt{2}X_2(t)dW(t),$$

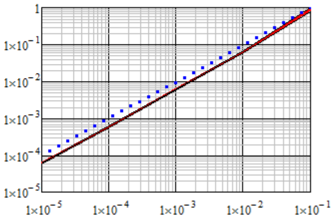
$$dX_1(t) = X_2(t)dt + \sqrt{2}X_2(t) \circ dW(t),$$

$$dX_2(t) = \big(-X_1(t) - X_2(t)\big)dt - \sqrt{2}X_1(t)dW(t),$$

$$dX_2(t) = -X_1(t)dt - \sqrt{2}X_1(t) \circ dW(t).$$

Examples: Elliptic cylinder





Rosenbrock type method, $[t_0, T] = [0, 1]$, $h = 0.001$.

Examples: Hyperbolic cylinder

Manifold equation (invariant): $M(t, x_1, x_2) = x_1 x_2 = C = \text{const}$.

The corresponding drift and diffusion coefficients are

$$f_1(t, x_1, x_2) = -\frac{x_1 q_1(t, x_1, x_2)}{x_1 q_2(t, x_1, x_2) - x_2 q_3(t, x_1, x_2)} + \frac{x_1}{2} q_0^2(t, x_1, x_2),$$

$$f_2(t, x_1, x_2) = \frac{x_2 q_1(t, x_1, x_2)}{x_1 q_2(t, x_1, x_2) - x_2 q_3(t, x_1, x_2)} + \frac{x_2}{2} q_0^2(t, x_1, x_2),$$

$$\sigma_1(t, x_1, x_2) = x_1 q_0(t, x_1, x_2), \quad \sigma_2(t, x_1, x_2) = -x_2 q_0(t, x_1, x_2),$$

where $q_0(t, x_1, x_2) = q_1(t, x_1, x_2) = 1$, $q_2(t, x_1, x_2) = 2/x_1$, and $q_3(t, x_1, x_2) = 1/x_2$.

Itô and Stratonovich SDEs:

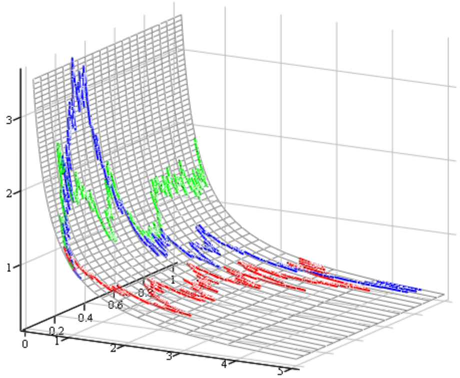
$$dX_1(t) = -\tfrac{1}{2} X_1(t)dt + X_1(t)dW(t),$$

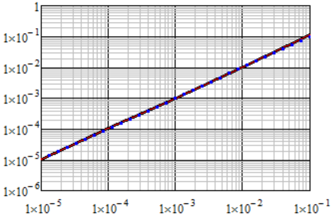
$$dX_1(t) = -X_1(t)dt + X_1(t) \circ dW(t),$$

$$dX_2(t) = \tfrac{3}{2} X_2(t)dt - X_2(t)dW(t),$$

$$dX_2(t) = X_2(t)dt - X_2(t) \circ dW(t).$$

Examples: Hyperbolic cylinder





Rosenbrock type method, $[t_0, T] = [0, 1]$, $h = 0.001$.

Examples: Parabolic cylinder

Manifold equation (invariant): $M(t, x_1, x_2) = x_2 - x_1^2 = C = \text{const}.$

The corresponding drift and diffusion coefficients are

$$f_1(t, x_1, x_2) = -\frac{q_1(t, x_1, x_2)}{2x_1 q_3(t, x_1, x_2) + q_2(t, x_1, x_2)},$$

$$f_2(t, x_1, x_2) = -\frac{2x_1 q_1(t, x_1, x_2)}{2x_1 q_3(t, x_1, x_2) + q_2(t, x_1, x_2)} + q_0^2(t, x_1, x_2),$$

$$\sigma_1(t, x_1, x_2) = q_0(t, x_1, x_2), \quad \sigma_2(t, x_1, x_2) = 2x_1 q_0(t, x_1, x_2),$$

where $q_0(t, x_1, x_2) = q_1(t, x_1, x_2) = 1$, $q_2(t, x_1, x_2) = -1$,
and $q_3(t, x_1, x_2) = 1/x_1$.

Itô and Stratonovich SDEs:

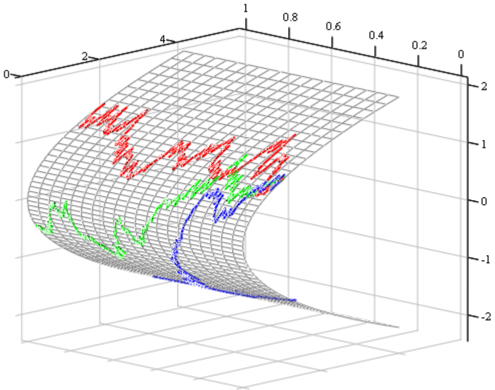
$$dX_1(t) = -dt + dW(t),$$

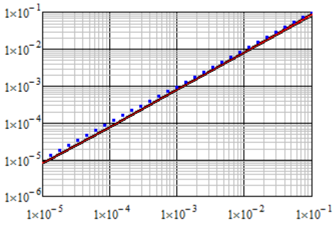
$$dX_1(t) = -dt + dW(t),$$

$$dX_2(t) = \big(1 - 2X_1(t)\big)dt + 2X_1(t)dW(t),$$

$$dX_2(t) = -2X_1(t)dt + 2X_1(t) \circ dW(t).$$

Examples: Parabolic cylinder





Rosenbrock type method, $[t_0, T] = [0, 1]$, $h = 0.001$.

Results of numerical experiments

1. Numerical methods for Itô SDEs:
   - Euler–Maruyama method ($p = 0.5$),
   - Milstein method ($p = 1.0$),
   - Platen method ($p = 1.0$),
   - Artemiev method ($p = 0.5$).

2. Numerical methods for Stratonovich SDEs:
   - Heun method ($p = 1.0$),
   - Derivative-free Heun method ($p = 1.0$),
   - Artemiev method ($p = 1.0$),
   - Averina method ($p = 1.0$).

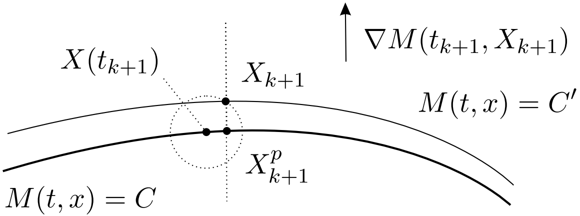
Projection method

Correction step:

$$X_{k+1}^{p} = X_{k+1} + \alpha(t_{k+1}, X_{k+1}) \nabla M(t_{k+1}, X_{k+1}), \quad X_{k+1} := X_{k+1}^{p},$$

where

$$\alpha(t_{k+1}, X_{k+1})\colon M(t_{k+1}, X_{k+1}^{p}) = C \quad \big(M(t_{k+1}, X_{k+1}) \neq C\big).$$



Projection method

Elliptic cylinder:

$$\alpha(t,X) = \frac{1}{2}\left(\sqrt{\frac{C}{M(t,X)}} - 1\right).$$

Hyperbolic cylinder:

$$\alpha(t,X) = \frac{-|X|^2 + \sqrt{|X|^4 - 4M(t,X)(M(t,X) - C)}}{2M(t,X)}.$$

Parabolic cylinder:

$$\alpha(t,X) = \frac{1 + 4X_0^2 - \sqrt{8X_0^2(2X_1 + 1 - 2C) + 1}}{8X_0^2}.$$

Numerical methods for Itô SDEs

1. The Euler–Maruyama method[7]:

$$X_{k+1} = X_k + hf(t_k, X_k) + \sqrt{h}\sigma(t_k, X_k)\Delta W_k.$$

2. The Milstein method[8,9]:

$$\begin{aligned} X_{k+1} = X_k + hf(t_k, X_k) + \sqrt{h}\sigma(t_k, X_k)\Delta W_k \\ + \frac{h}{2}\frac{\partial\sigma(t_k, X_k)}{\partial x}\sigma(t_k, X_k)(\Delta W_k^2 - 1). \end{aligned}$$

---

[7] G. Maruyama, "Continuous Markov processes and stochastic equations," *Rend. Circolo Math. Palermo*, vol. 2, no. 4, pp. 48–90, 1955.

[8] G. N. Milstein, "Approximate integration of stochastic differential equations," *Theory Probab. Appl.*, vol. 19, no. 3, pp. 557–562, 1974.

[9] G. N. Milstein, M. V. Tretyakov, *Stochastic Numerics for Mathematical Physics*, Springer, 2004.

Numerical methods for Itô SDEs

3. The Platen method[10]:

$$X_{k+1} = X_k + hf(t_k, X_k) + \frac{\sqrt{h}}{2}\big(\sigma(t_k, X_k^p) - \sigma(t_k, X_k)\big)(\Delta W_k^2 - 1),$$
$$X_k^p = X_k + \sqrt{h}\sigma(t_k, X_k)\Delta W_k.$$

4. The Artemiev method[11,12]:

$$X_{k+1} = X_k + \left[I - \frac{h}{2}\frac{\partial f(t_k, X_k)}{\partial x}\right]^{-1}\left[hf(t_k, X_k) + \sqrt{h}\sigma(t_k, X_k)\Delta W_k\right].$$

---

[10] P. E. Kloeden, E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, 1999.

[11] S. S. Artemiev, T. A. Averina, *Numerical Analysis of Systems of Ordinary and Stochastic Differential Equations*, VSP, 1997.

[12] T. A. Averina, S. S. Artemiev, "A new family of numerical methods for solving stochastic differential equations," *Soviet Math. Dokl.*, vol. 33, no. 3, pp. 736–738, 1986.

Numerical methods for Stratonovich SDEs

1. The Heun method[13]:

$$X_{k+1} = X_k + \frac{h}{2}\left(a(t_k, X_k) + a(t_{k+1}, X_k^p)\right) + \frac{\sqrt{h}}{2}\left(\sigma(t_k, X_k) + \sigma(t_{k+1}, X_k^p)\right)\Delta W_k,$$

$$X_k^p = X_k + h\left[a(t_k, X_k) + \frac{1}{2}\frac{\partial\sigma(t_k, X_k)}{\partial x}\sigma(t_k, X_k)\right] + \sqrt{h}\sigma(t_k, X_k)\Delta W_k.$$

2. The derivative-free Heun method[14,15]:

$$X_{k+1} = X_k + \frac{h}{2}\left(a(t_k, X_k) + a(t_{k+1}, X_k^p)\right) + \frac{\sqrt{h}}{2}\left(\sigma(t_k, X_k) + \sigma(t_{k+1}, X_k^p)\right)\Delta W_k,$$

$$X_k^p = X_k + ha(t_k, X_k) + \sqrt{h}\sigma(t_k, X_k)\Delta W_k.$$

---

[13] K. Burrage, P. M. Burrage, T. Tian, "Numerical methods for strong solutions of stochastic differential equations: an overview," *Proc. R. Soc. Lond. A*, vol. 460, no. 2041, pp. 373–402, 2004.

[14] P. E. Kloeden, R. A. Pearson, "The numerical solution of stochastic differential equations," *J. Aust. Math. Soc. B*, vol. 20, pp. 8–12, 1977.

[15] N. N. Nikitin, V. D. Razevig, "Digital simulation of stochastic differential equations and error estimates," *USSR Comput. Math. Math. Phys.*, vol. 18, no. 1, pp. 102–113, 1978.

Numerical methods for Stratonovich SDEs

3. The Artemiev method[16,17]:

$$X_{k+1} = X_k + \left[I - \frac{h}{2}\frac{\partial a(t_k, X_k)}{\partial x}\right]^{-1}\left[ha(t_k, X_k) + \sqrt{h}\sigma(t_k, X_k)\Delta W_k \right.$$
$$\left. + \frac{h}{2}\frac{\partial \sigma(t_k, X_k)}{\partial x}\sigma(t_k, X_k)\Delta W_k^2\right].$$

4. The Averina method[18]:

$$X_{k+1} = X_k + \frac{h}{2}\left[I - \frac{h}{2}\frac{\partial a(t_k, X_k)}{\partial x}\right]^{-1}\left[a(t_k, X_k) + a(t_k, X_k^p)\right]$$
$$+ \frac{\sqrt{h}}{2}\left(\sigma(t_k, X_k) + \sigma(t_k, X_k^p)\right)\Delta W_k, \quad X_k^p = X_k + \sqrt{h}\sigma(t_k, X_k)\Delta W_k.$$

---

[16] S. S. Artemiev, T. A. Averina, *Numerical Analysis of Systems of Ordinary and Stochastic Differential Equations*, VSP, 1997.

[17] T. A. Averina, S. S. Artemiev, “A new family of numerical methods for solving stochastic differential equations,” *Soviet Math. Dokl.*, vol. 33, no. 3, pp. 736–738, 1986.

[18] T. A. Averina, *Construction of Statistical Modeling Algorithms for Systems with Random Structure*, Novosibirsk State University, 2015.