Group properties of evolutionary integro-differential equation*

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The paper deals with an evolutionary integro-differential equation describing nonlinear waves. Particular choice of the kernel in the integral leads to well-known equations such as the Khokhlov-Zabolotskaya equation, the Kadomtsev-Petviashvili equation and others. Since solutions of these equations describe many physical phenomena, analysis of the general model studied in the paper equation is important. One of the methods for obtaining solutions differential equations is provided by the Lie group analysis. However, this method is not applicable to integro-differential equations. Therefore we discuss new approaches developed in modern group analysis and apply them to the general model considered in the present paper. Reduced equations and exact solutions are also presented.

The nonlinear integro-differential evolution equation

$$\begin{array}{l} (u_x - uu_t - w_{tt})_t = u_{yy} + u_{zz}\,, \\ w = \int_0^\infty K(s)\; u(t-s)\, ds \end{array} . \tag{1}$$

considered in the present paper is not an exotic model (see review in [1]). It encapsulates numerous mathematical models formulated by differential evolution equations and differs from them significantly not only in its form, but mostly due to its physical content meaning. Namely, any dispersion (frequency-dependent phase velocity) must be strongly connected with frequency-dependent absorption. Such connection follows from the causality principle. For example, waves having infinite velocities of propagation which are allowed by differential equations of Burgers and Korteweg-de Vries type must disappear on their way, since otherwise a cause appears at a certain point later than its effect. The causality principle is given in physical models by integral Kramers-Kronig relations. Consequently, a consistent model must contain integral terms, in other words, be represented in an integro-differential form. Though this conclusion is well known, mathematical models described by purely differential evolution equations have been widely accepted in the nonlinear wave physics due to their simplicity compare to the integro-differential models. It seems that the consistent integro-differential nonlinear models will meet more applications in future.

*This research was partially supported by BTH, RFBR and Presidential Program for Leading Scientific School. The work of SVM was supported by the NRCT (SUT).

The present paper provides a first step in application of the Lie group analysis to Equation (1). The approach used in this paper is described in [2]. The analysis of the determining equation for the integro-differential equation allows, in particular, to single out a class of kernels used for deriving mathematical models in medical applications of ultrasound [3].

Note that for particular kernels the integro-differential equation (1) becomes a partial differential equation or a delay partial differential equation. In these cases the complete group classification of equation (1) can be obtained. In the case of partial differential equations the classical group analysis is used. For delay partial differential equations the analysis developed in [4] and described in [2] is applied. A complete study of particular cases is given in the paper. This provides a new result in the application of the group analysis method to partial and delay partial differential equations.

Along with admitted Lie groups, representations of exact solutions and reduced equations are constructed in the paper. A complete solution and a physical interpretation of some of them is presented.

We hope that more results will be obtained in future by applying the above approach for solving concrete models of physical significance as well as for new mathematical developments. In particular, it is interesting to make the preliminary group classification of exceptional kernels by applying the method of an a priory use of symmetries [5] to the integro-differential equations of the form (1).

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