# Wave Equations with $p(x,t)-$ Laplacian and Damping Term : Existence and Blow-up

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We study the Dirichlet problem

$$u_{tt} = div\left(a(x,t)\,|\nabla u|^{p(x,t)-2}\,\nabla u\right) + \alpha \triangle u_t + b(x,t)|u|^{\sigma(x,t)-2}u, \quad (x,t) \in Q_T,$$
$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega,$$
$$u|_{\Gamma_T} = 0, \Gamma_T = \partial\Omega \times (0,T).$$

Under suitable condition on the data, we prove local and global existence theorems and study the finite time blow-up of the solutions. The analysis relies on the methods developed in [1,2,3].

## 1. Statement of the problem

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with Lipschitz-continuous boundary $\Gamma$ and $Q_T = \Omega \times (0,T]$. We consider the following boundary value problem

$$\begin{gathered} u_{tt} = div\left(a\,|\nabla u|^{p(x,t)-2}\,\nabla u\right) + \alpha \triangle u_t + b\,|u|^{\sigma(x,t)-2}\,u + f, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \\ u|_{\Gamma_T} = 0,\ \Gamma_T = \partial\Omega, \times(0,T) \end{gathered} \tag{1}$$

with $\alpha = const > 0$. The coefficients $a(x,t),\ b(x,t)$, exponents $p(x,t), \sigma(x,t)$ and the source term $f(x,t)$ are given functions of their arguments satisfying

$$0 < a_- \le a(x,t) \le a_+ < \infty,\ |b(x,t)| \le b_+ < \infty, \tag{2}$$

$$1 < p_- \le p(x,t) \le p_+ < \infty, 1 < \sigma_- \le \sigma(x,t) \le \sigma_+ < \infty, \tag{3}$$

$$f \in L^2(Q_T),\ u_1 \in L^2(\Omega),\ u_0 \in L^2(\Omega) \cap L^{\sigma(\cdot,0)}(\Omega) \cap W^{1,p(\cdot,0)}(\Omega). \tag{4}$$

Problem (1) appears in models of nonlinear viscoelasticity (see [4,5,6]).The local and global existence and blow up for hyperbolic equations of the type (1) with constant exponents of nonlinearity have been studied in many papers-see, e.g.,[4,7].However, only papers [8,9]are devoted to the study of hyperbolic equations of the type (1) with variable nonlinearities. In the present communication, we discuss how the variable character of nonlinearity influences the existence and blow-up theory for the EDPs of the type [1].

## 2. Existence theorem

Let $W_0 = W_0(Q_T)$ be a set of the functions $u(x,t)$ such that

$$\nabla u_t \in L^2(Q_T), \; u(\cdot,t) \in W_0^{1,1}(\Omega) \; a.e. \text{ in } [0,T],$$

$$\left(u_t, |\nabla u|^{p/2}, |u|^{\sigma/2}\right) \in L^\infty\left(0,T; L^2(\Omega)\right).$$

We introduce the norm in $W_0(Q_T)$ by

$$\|u\|_{W_0} = \|u\|_{L^2(Q_T)} + \|u\|_{L^{\sigma(\cdot)}(Q_T)} + \|u_t\|_{L^\infty(0,T;\; L^2(\Omega))} +$$

$$+ \|\nabla u\|_{L^\infty(0,T;\; L^2(\Omega))} + \|\nabla u_t\|_{L^2(Q_T)} + \|\nabla u\|_{L^{p(\cdot)}(Q_T)}.$$

Let us assume that

$$|p(x,t) - p(y,\tau)| \le \omega(|x-y| + |t-\tau|), \; \overline{\lim_{s\to+0}}\, \omega(s) \ln \frac{1}{s} \le C < \infty. \tag{5}$$

**Definition 1.** *A function $u$: $\Omega_T \to \mathbf{R}$ is called a weak solution to problem (1) if:*

- *(i)* $u \in L^\infty(0,T; W^{1,p_+}(\Omega)) \cap W^{1,\infty}(0,T; L^2(\Omega))$,
- *(ii)*
$$u(\cdot,t) \rightharpoonup u_0 \;\; in \; L^2(\Omega) \cap W^{1,P}(\Omega), \;\; u_t(\cdot,t) \rightharpoonup u_1 \;\; in \; L^2(\Omega), \tag{6}$$
- *(iii)* $\forall \varphi \in C^\infty\left(0,T; C_0^\infty(\Omega)\right), \; \varphi(x,T) = 0, \; \Omega \in x$ *the following integral identity holds*

$$\int_{Q_T} \left(-u_t\varphi_t + \left(a\, |\nabla u|^{p(x)-2} \nabla u + \alpha \nabla u_t\right) \cdot \nabla\varphi - b(x,t)\, |u|^{\sigma(x,t)-2}\, u\varphi\right) \tag{7}$$

$$= \int_\Omega u_1 \varphi(\cdot,0) + \int_{Q_T} f\varphi.$$

The proof of existence theorem is based on modified methods of Galerkin and method of monotonicity and on a priori estimates.

### 2.1. Energy relation

The energy function

$$E(t) = E\left[u, u_t\right] = \int_\Omega \left[\frac{|u_t|^2}{2} + a(\cdot,t)\frac{|\nabla u|^{p(\cdot,t)}}{p(\cdot,t)} - b(\cdot,t)\frac{|u|^{\sigma(\cdot,t)}}{\sigma(\cdot,t)}\right],$$

satisfies the energy relation

$$E'(t) + \alpha \int_\Omega |\nabla u_t(\cdot,t)|^2 = \Lambda,$$

in which

$$\Lambda(t) = \Lambda_1 + \int_\Omega \left(\frac{b|u\,|^\sigma}{\sigma^2}\left(1 - \sigma^2 \ln|u\,|\right)\sigma_t\right) + \int_\Omega -b_t\frac{|u|^\sigma}{\sigma} + \int_\Omega f u_t,$$

$$\Lambda_1 = \int_\Omega \left[a_t \frac{|\nabla u\,|^p}{p} + a\frac{|\nabla u\,|^p}{p^2}\left(-1 + p^2 \ln|\nabla u\,|\right)p_t\right].$$

## 2.2. A priori estimates

**Lemma 1.** *(Global estimates) Let us assume that conditions (2)-(4) are fulfilled and*

$$|a_t| \leq C_a, |b_t| \leq C_b, \ p_t \leq 0, \ \sigma_t \geq 0,$$

$$0 \leq b_- \leq -b(x,t) \leq b_+ \leq \infty, \ or \ (\sigma_+ \leq \max[2, p_- - \delta]),$$

$$p_t \leq 0, \ \sigma_t \leq 0, |p_t| \leq C_p, |\sigma_t| \leq C_\sigma.$$

*Then for any finite $T > 0$ any $t \in [0, T]$*

$$\Psi(t) = \int_\Omega \left[ |u_t|^2 + |\nabla u|^{p(\cdot)} + |u|^{\sigma(\cdot)} \right] + +\alpha \int_0^t \int_\Omega |\nabla u_t|^2 \leq C \tag{8}$$

*with a constant $C\left(T, \|f\|^2_{2,Q_T}, E(0)\right)$.*

**Lemma 2.** *(Estimate for small time $T$). Assume that*

$$0 \leq a_- \leq a(x,t) \leq a_+ \leq \infty, \ |a_t| \leq C_a, \ p_t \leq 0, \sigma_t \geq 0,$$

$$0 \leq b_- \leq b(x,t) = b(x,t) \leq b_+ \leq \infty, \ |b_t| \leq C_b,$$

$$p_t \leq 0, \ \sigma_t \leq 0, |p_t| \leq C_p, |\sigma_t| \leq C_\sigma,$$

$$2 < \sigma_- \leq \sigma_+ < \frac{n+2}{n} p_- \leq \frac{np_-}{n - p_-} < \infty.$$

*Then there exists a small $T_0 > 0$, such that estimate (8) be valid for $t \leq T_{0.}$*

**Theorem 1.** *(a) Let condition (5)and the conditions of Lemma 1 and be fulfilled. Then for any finite $T > 0$ problem (1) has at least one weak solution $u \in W_0$ in the sense of Definition 1.*

*(b) If condition (5) and the conditions of Lemma 2 are fulfilled, then there exists a local in time solution $u \in W_0$ for $t \in [0, T_0]0$.*

# 3. Blow up

Let us introduce the function $G(t) = \|u(t)\|^2$ and assume that

$$G'(0) = 2\langle u(0), u_t(0)\rangle \equiv 2\langle u_0, u_1\rangle > 0. \tag{9}$$

Under the Theorem 1 (b)(local in time existence) the following global nonexistence result is true.

**Theorem 2.** *Theorem 2. Let the conditions of Lemma 2 be valid. If $E(0) < 0$, then the solution of the problem (1) blows up ( in the sense that $G(t) = \|u(t)\|^2$ becomes unbounded on the finite interval $(0, T)$) with $T = \frac{2\|u_0\|^2}{(\lambda-2)\langle u_0, u_1\rangle}$.*

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