Adaptive modelling of surface water flows with wetting and drying over complex bottom topographies

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The adaptive modelling of surface water flows over complex bottom topographies, taking into account possible processes of wetting and drying is considered. This work continues our investigations [1] within the project "Adaptive Hydrological Modelling with Application in Water Industry"[2] of the Federal Ministry of Education and Research of Germany. The 2D shallow water model is used, including bottom friction and a viscosity term. The implementation is based on the DUNE-FEM module - a modular toolbox for solving PDEs with grid-based methods [3, 4]. The problem is numerically solved by the Runge-Kutta discontinuous Galerkin method [5]. A well-balancing method [1] is used, based on a reformulation of the topography source term in the balance law for the discharge. The wetting-drying treatment, based on the ideas of [6], is incorporated into the model. The newly developed limiter [7] is used for the method stabilization. The code is validated on several test problems with known exact solutions and tested on few more complex problems with source, bottom friction and diffusion terms.

1. Runge-Kutta discontinuous Galerkin method for shallow water equations

Let us provide the formulation of the Runge-Kutta Discontinuous Galerkin method [5] for the evolution equations of a very general form:

$$\partial_t \mathbf{u}(t,\cdot) = \mathcal{L}[\mathbf{u}(t,\cdot)](\cdot) \qquad \text{in } ([0,T)\times\Omega) \subset (I\!R \times I\!R^d),\ d \in \{1,2,3\}, \tag{1}$$

with the spatial operator $\mathcal{L}[\mathbf{v}] = S(\mathbf{v}) - \nabla \cdot \big(F(\mathbf{v}) - G(\mathbf{v})\big)$, where $\mathbf{v} : \Omega \to \Psi \subseteq I\!R^r$ belongs to some suitable function space V, Ψ is the set of states for a given problem, $S(\mathbf{v})$ is a source term function, $F(\mathbf{v})$ is the convective analytical flux function, $G(\mathbf{v})$ is the eddy viscous analytical flux function. The appropriate definition of $\mathbf{u}$, S, F and G gives the 2D shallow water equations:

$$\mathbf{u} = \begin{pmatrix} h \\ hu \\ hv \end{pmatrix}, \quad F(\mathbf{u}) = \begin{pmatrix} hu & hv \\ hu^2 + \frac{1}{2}gh^2 & huv \\ huv & hv^2 + \frac{1}{2}gh^2 \end{pmatrix}, \quad G(\mathbf{u}) = \begin{pmatrix} 0 & 0 \\ \nu h \frac{\partial u}{\partial x} & \nu h \frac{\partial u}{\partial y} \\ \nu h \frac{\partial v}{\partial x} & \nu h \frac{\partial v}{\partial y} \end{pmatrix}, \tag{2}$$

$$S(\mathbf{u}) = S_h(\mathbf{u}) + S_{bs}(\mathbf{u}) + S_{bf}(\mathbf{u}) = \begin{pmatrix} S_h \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -gh\frac{\partial b}{\partial x} \\ -gh\frac{\partial b}{\partial y} \end{pmatrix} + \begin{pmatrix} 0 \\ -ghI_{R_x} \\ -ghI_{R_y} \end{pmatrix}. \quad (3)$$

$\mathbf{u} = (u, v)$ is the velocity vector, $h = h(t, \mathbf{x})$ is the water depth, measured from the bottom positive upwards, $b = b(\mathbf{x})$ is the bottom height, measured from the reference level positive upwards. The total water height $H = H(t, \mathbf{x}) = h(t, \mathbf{x}) + b(\mathbf{x})$ is also measured from the reference level positive upwards, g is the gravity acceleration. $\nu = 0.6\frac{1}{h^{1/3}}\sqrt{g}n|\mathbf{u}|h$ is the eddy viscosity coefficient, where n is the Manning coefficient.

The source term S_h takes into account sources and sinks due to the coupling with the groundwater flow. The bottom slope term S_{bs} takes into account the bottom topography. The bottom friction term S_{bf} contains the head loss I_R, which is evaluated using the Darcy-Weisbach formula $I_R = \frac{\lambda \mathbf{u}|\mathbf{u}|}{2gD}$, where $\lambda = 6.34\frac{2gn^2}{D^{1/3}}$ is the friction factor, D is the hydraulic diameter, for the 2D SWE $D = 4h$, n is the Manning coefficient. Additional forces, such as bottom friction forces, Coriolis forces, tidal potential forces, wind surface stresses, can be added. The proper initial and boundary conditions [8] have to be added to the equations.

For the spatial discretization, a discrete operator $\mathcal{L}_h$ is defined, mapping a discrete function space V_h. We choose $V_h := \{\varphi : \Omega_h \to I\!R \in L^2(\Omega_h) \mid \varphi|_E \in \mathcal{P}_p(E) \ \ \forall E \in \mathcal{T}_h\}$, where $\mathcal{P}_p(E)$ is the set of all polynomials of an order up p. The set $\Omega_h \subseteq \Omega$ is a polygonal approximation of the domain Ω which is partitioned by a tesselation $\mathcal{T}_h$ in the sense of the grid definition from [9]. The discrete operator is given by

$$\begin{aligned} \int_{\Omega_h} \mathcal{L}_h[u_h]\varphi\, d\mathbf{x} &= \sum_{E\in\mathcal{T}_h} \int_E S(u_h)\varphi\, d\mathbf{x} + \sum_{E\in\mathcal{T}_h} \int_E (F(u_h) + G(u_h) \cdot \nabla\varphi\, d\mathbf{x} - \\ &- \sum_{E\in\mathcal{T}_h} \int_{\partial E} \varphi\, F_h(u_h^+, u_h^-, \ldots) + G_h(u_h^+, u_h^-, \ldots)\, d\sigma \qquad \forall \varphi \in V_h. \quad (4) \end{aligned}$$

Here u_h^+ and u_h^- are the values of the function u_h on both sides of the element interface and $F_h(u, v, \mathbf{x}), G_h(u, v, \mathbf{x}) : V \times V \times I\!R^d \to I\!R^d$ are numerical flux functions. The discontinuous Galerkin method is completely described by the functions S and F, G and the numerical fluxes F_h, G_h and the space V_h. In this work, the Local-Lax-Friedrichs flux function ([10]) is used as the numerical flux F_h for the advection part. For the diffusion flux G_h we use the CDG2 method derived and analysed in [11]. This method is an efficient variation of the compact Discontinuous Galerkin method first suggested for elliptic problems in [12].

The RK-DG method is in general not stable for non-linear convection dominated problems where steep gradients or strong shocks might appear, thus, we use the stabilization mechanism [7], which was expanded in the context of our project to a specific criterion such as guaranteeing the conservation of non-negative water level. A stabilized discrete operator is constructed by concatenation of the DG operator $\mathcal{L}_h$ and a stabilization operator Π_h: $\tilde{\mathcal{L}}_h[v_h(t, \cdot)] := (\mathcal{L}_h \circ \Pi_h)[v_h(t, \cdot)]$. For the time discretization an operator $\bar{\Pi}_h$ is used for the initial data on each time step. The construction of $\Pi_h, \bar{\Pi}_h$ is given in [7].

The space discretization leads to a system of ODEs $\frac{d}{dt}u_h(t) = \tilde{\mathcal{L}}_h[u_h(t)]$ for the coefficients defining $u_h(t)$. An explicit Runge-Kutta method is used with the order $k = p + 1$, where p is the polynomial degree of the basis functions used to construct the space V_h.

2. Well-balancing

There have been many suggestions for the construction of well-balanced schemes in recent years, some applied to the RK-DG scheme (see, for example, [13]). We suggest a new method here, which is easy to implement in the DG framework as implemented in DUNE-FEM [14].

Our method is based on a simple reformulation of the topography source term in the shallow water model. We focus on the balance law for the discharge $q = (hu, hv)$ and neglect the viscous flux and the source term due to the ground water coupling since they do not contribute to the problem of well balancing. In its original form this equation reads $\partial_t q + \nabla \cdot F_q = S_q$ where F_q is the flux and $S_q = -gh\nabla b$ (see (2)). The most important requirement for a well-balanced scheme is to preserve a "lake at rest", i.e., if $u = 0$, $v = 0$, and $h = C - b$ with some constant C, then $F_q = S_q$ should be satisfied also on a discrete level. Inserting $u = 0$ and replacing b with $C - h$ in the equation for the discharge, the balance of flux and source term is given by $\frac{1}{2}\nabla h^2 - h\nabla h = 0$.

We start by rewriting the topology source term: $S_q = -\frac{1}{2}gh\nabla b - \frac{1}{2}\nabla \cdot (ghb) + \frac{1}{2}gb\nabla h$. Moving the divergence term to the left hand side and treating the topography b as an additional unknown we arrive at:

$$\partial_t h + \nabla \cdot q = 0, \tag{5}$$

$$\partial_t b = 0, \tag{6}$$

$$\partial_t q + \nabla \cdot (F_q + \frac{g}{2}hb) = -\frac{g}{2}(h\nabla b - b\nabla h). \tag{7}$$

Now in the DG context, the non-conservative products on the right hand side can be discretized using the approach from [15], treating them as a source term together with a measure on the boundary. We focus on the spatial discretization of the equation for the discharge q, now denoting with h, b the discrete solutions defined by the DG framework, i.e., we assume that h, b are piecewise polynomial functions, where the topography is projected into the same discrete space in which h is defined. The spatial discretization on a single element T of the grid is given by:

$$\begin{aligned} L_T[h, b, q] \; := \; & \int_T (F_q + \frac{g}{2}bh) \cdot \nabla\varphi - \int_T \frac{g}{2}(h\nabla b - b\nabla h)\varphi - \\ & \int_{\partial T} (\widehat{F_q} + \frac{g}{2}\widehat{hb}n)\,\varphi + \frac{g}{2}(\widetilde{h\varphi}[b] - \widetilde{b\varphi}[h]), \varphi \in V_h. \end{aligned} \tag{8}$$

Here we use the abbreviation $\widehat{F_q}$ and $\widehat{hb}$ to denote numerical flux functions, approximating fluxes in normal direction over the cell boundaries (e.g. a Lax-Friedrichs flux for F_q and an averaging for hb). With $\widetilde{h\varphi}$ and $\widetilde{b\varphi}$ we denote some suitable averages and $[b] = (b^+ - b_T)n$, $[h] = (h^+ - h_T)n$ are the jumps of the (possibly) discontinuous discrete functions b, h in the normal direction n; h_T, b_T denote the discrete functions on the element T and h^+, b^+ the values on neighbouring elements.

The test function φ is assumed to have support only on element T (which means that $\varphi^+ = 0$). It turns out that a very simple averaging process leads to good results, i.e., $\widetilde{h\varphi} = \widetilde{h}\widetilde{\varphi} = \frac{1}{4}\varphi(h_T + h^+)$ and $\widetilde{b\varphi} = \frac{1}{4}\varphi(b_T + b^+)$; but note that the method will work with a much more general choice for the averaging procedure. To achieve well-balancing we focus on the case $q \equiv 0$ and $h + b = C$ on each element, which characterizes a lake at rest. Note that both b and h are assumed to belong to the same discrete space, so that $h + b = C$ holds on the discrete level. In this scenario $[b] = -[h]$ and the only assumption we make on the averaging is that under these circumstances $\widetilde{b\varphi} = \frac{1}{2}C\varphi - \widetilde{h\varphi}$. The only assumption we make on the underlying scheme is that in this situation $\widehat{F_q} = \frac{g}{2}\widehat{h^2}n$. We thus arrive at:

$$\begin{aligned} L_T[h,b,q] &= \int_T \frac{g}{2}\big(h^2 + (C-h)h\big)\cdot\nabla\varphi - \int_T \frac{g}{2}(-h\nabla h - (C-h)\nabla h)\varphi - \\ &\quad - \int_{\partial T}\frac{g}{2}(\widehat{h^2} + \widehat{hb})n\varphi + \frac{g}{2}(-\widetilde{h\varphi}[h] - (\frac{1}{2}C\varphi - \widetilde{h\varphi})[h]) = \\ &= \frac{g}{2}C\int_T \big(h\nabla\varphi + \nabla h\varphi\big) - \frac{g}{2}\int_{\partial T}(\widehat{h^2} + \widehat{hb} - \frac{1}{2}C(h^+ - h_T))n\varphi = \\ &= -\frac{g}{2}\int_{\partial T}(\widehat{h^2} + \widehat{hb} - \frac{1}{2}C(h^+ - h_T) - Ch_T)n\varphi = \\ &= -\frac{g}{2}\int_{\partial T}(\widehat{h^2} + \widehat{hb} - C\frac{1}{2}(h^+ + h_T))n\varphi\ . \end{aligned} \tag{9}$$

For well-balancing we need $L_T[h,b,q] = 0$, which is satisfied if we choose $\widehat{hb} = -\widehat{h^2} + C\frac{1}{2}(h^+ + h_T)$. Since $b = C - h$ we have $hb = Ch - h^2$ so that this is a reasonable assumption. If, for example, F_q is given by the Lax-Friedrichs scheme, we have (due to $q = 0$) $\widehat{F_q} = \frac{g}{4}(h_T^2 + (h^+)^2)$ so that $\widehat{h^2} = \frac{1}{2}(h_T^2 + (h^+)^2)$. With the simple choice of $\widehat{hb} = \frac{1}{2}(h_T b_T + h^+ b^+) = \frac{1}{2}(Ch_T - h_T^2 + Ch^+ - (h^+)^2) = -\widehat{h^2} + C\frac{1}{2}(h_T + h^+)$ so that the modified scheme is in fact well-balanced.

3. Wetting and Drying Treatment

The wetting and drying treatment used in the current work is based on the algorithm [6], belonging to the thin slot algorithms. For more details and further references on thin slot algorithms and wetting-drying treatments in general see [1]. According to [6] the so-called wet-or-dry status flags are initially set, marking each element E either “wet” (flag = 1) or “dry” (flag = 0), and then updated at each k-th Runge-Kutta intermediate stage. The stability condition for Δt is based on the consideration of the balance between the mass in the element and the outgoing discharge through its boundary. Every intermediate RK-stage must preserve the positivity of a mean water depth in every cell. If this stability condition shows that flows empties an element through a considered boundary, then according to [6] the so-called “reflection numerical flux” is evaluated, which prohibits mass transfer through a boundary. If a considered boundary is an interface between two dry elements, then the reflection numerical flux is used without checking a stability condition in order to do not introduce an artificial mass exchange between dry elements. In our work we don’t use this condition for Δt, but apply the reflection flux as soon as empting of an element is detected. This simpler condition nonetheless gives good numerical results.

We implement the requirement as in [6] to cancel the gravity terms in dry elements in order to avoid non-physical oscillations and to prevent dry elements from loosing their mass by using the above described wet-or-dry status flags in the mathematical model, setting $g = g_0$ in an element if it is “wet” and setting $g = 0$ if an element is “dry“. Thus, in our implementation the status flags belong to ”thin water layer“ shallow water mathematical model, which seems to us very logical.

After the positivity of the mean water depth is guaranteed in each element, the so-called Positive Depth operator [6] has to guarantee the positivity of water depth node-wise in each element.

The PD operator Π_h^{PD} is implemented together with the stabilization operator $\tilde{\mathcal{L}}_h[v_h(t,\cdot)] := (\mathcal{L}_h \circ \Pi_h \circ \Pi_h^{PD})[v_h(t,\cdot)]$. See [6] for the details on the PD operator.

Currently triangular elements and linear approximations of water depth and discharges are used in this algorithm. But due to the generic DUNE implementation it will possible to broaden on other elements and higher order approximations in the future. We can not prove the convergence of the developed algorithm, but the validation of the code shows only a small error between numerical result and known exact solution for the series of test problems. Also the realistic behaviour of the numerical solution for the problem with source/sink terms is shown. See the section 4 for the details.

4. Numerical Results

The described algorithm is implemented using **DUNE** – the **D**istributed and **U**nified **Nu**merics **E**nvironment [16] – a modular toolbox for solving partial differential equations with grid-based methods [16]. The DUNE module DUNE-SWE is currently being developed for numerical simulation of shallow water flows with well-balancing, limiting and taking into account wetting and drying processes. The Runge-Kutta Discontinuous Galerkin method for solving the shallow water equations with wetting and drying is implemented on the basis of DUNE-FEM module [4]. As both stabilization operator and wetting-drying treatment are based on the same concept of keeping some quantity realistic, we implement them within one pass [17].

The developed code is validated on problems with known exact solutions, where wetting-drying processes occur. The results show good correspondence of numerical results and the exact solution. See [1] for the details on the "lake at rest", "dam break problem", "drying Riemann problem" and "parabolic bowl problem". See also [1] for the numerical results of the "source-sink" problem, which was the first step towards the coupled modelling of surface and groundwater flows. Figure 1 shows the results of the numerical modelling of the flood wave in the part of the real river bed taking into account bottom friction and viscosity.

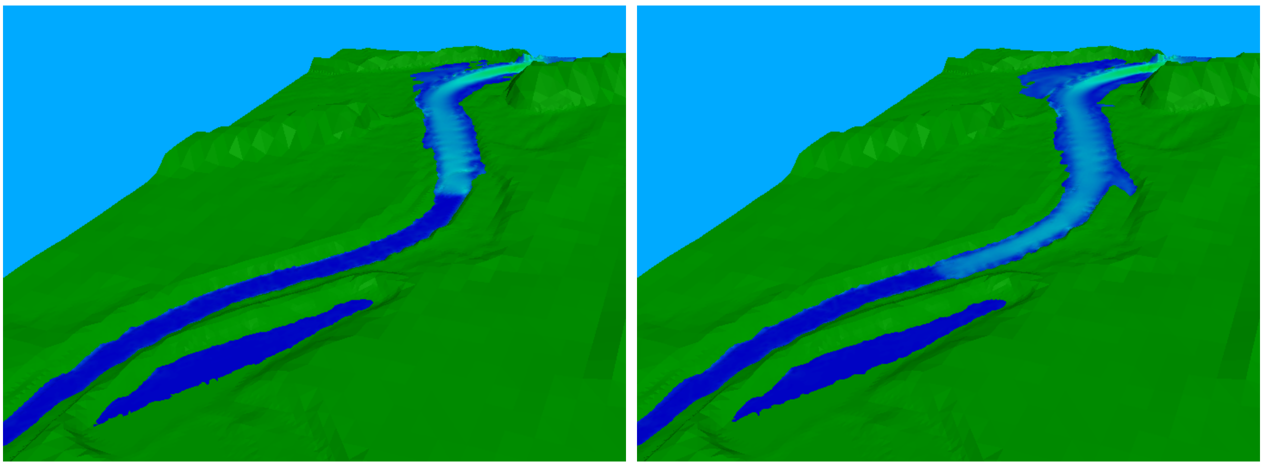

Рис. 1. Flood wave in the part of the real river bed.

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