Solvability of a generalized Buckley-Leverett model

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We propose a new mathematical modeling of the Buckley- Leverett system, which describes the two-phase flows in porous media. We prove the solvability of the initial-boundary value problem for a deduced model. In order to show the solvability result, we consider an approximated parabolic-elliptic system. Since the approximated solutions do not have **any type compactness property**, the limit transition is justified by the kinetic method. The main issue is to study a linear (kinetic) transport equation, instead of the nonlinear original system.

1. Introduction

A simultaneous motion of two immiscible incompressible liquids (e.g. water and oil) in a porous medium can be described by the famous Buckley-Leverett system

$$\partial_t u + \operatorname{div}\big(\mathbf{v}\ g(u)\big) = 0, \qquad \operatorname{div}(\mathbf{v}) = 0, \tag{1}$$

$$h(u)\mathbf{v} = -\nabla p, \tag{2}$$

where u, p and $\mathbf{v}$ are respectively the saturation, the pressure and the total velocity of the two-phase flow. The saturation dependent functions $h(u)$, $g(u)$ describe physical properties of the porous media. Equation (2) is Darcy's Law, being an empirical equation. The study of system (1)-(2) has a practical interest in connection with the planning and operation of oil wells, but brings some challenging mathematical questions.

Until nowadays the solvability of this system has not been shown. In order to pass this difficulty, the Buckley-Leverett system has been significantly simplified in many works, for instance see: Cordoba, Gancedo, Orive [1], Perepelitsa, Shelukhin [2]. Many authors have proposed interesting ideas, but most of them focused on the saturation equation (1): reducing the Buckley-Leverett system to an elliptic-parabolic partial differential system, here we address some of the important works on this subject: Antontsev, Kazikhov, Monakhov [3], Chen [4], Lenzinger, Schweizer [5] and further references cited therein.

In the present work we change the focus and put more attention to the equation of velocity. So we propose a generalized Darcy's law equation, which is no physically longer than the standard one cited above. In particular, for homogeneous, isotropic medium and one phase flow, we recall that, for very short time scales or high frequency oscillations, a time derivative of flux may be added to Darcy's law, which results in valid solutions at very small times

$$\tau\,\partial_t \mathbf{v} + h(u)\mathbf{v} = -\nabla p, \tag{3}$$

where $\tau > 0$ is a very small time constant. Another extension to the traditional form of Darcy's law is Brinkman´s term, which is used to account for transitional flow between boundaries

$$\nu\,\Delta \mathbf{v} + h(u)\mathbf{v} = -\nabla p, \tag{4}$$

where $\nu > 0$ is an effective viscosity. This correction term accounts for flow through medium where the grains of the media are porous themselves. In a porous media literature [6] the combination of (3) and (4) is known as Brinkman-Forchheimer's law

$$\tau\,\partial_t \mathbf{v} + \nu\,\Delta \mathbf{v} + h(u)\mathbf{v} = -\nabla p. \tag{5}$$

It is important to observe that, such generalized Darcy's laws (3), (4), (5) have been deduced also by the homogenization theory [7].

2. Generalized Buckley-Leverett model (1), (5)

Let $\Omega \subset \mathbb{R}^d$ (with $d = 1, 2$ or 3) be a bounded domain having a C^2- smooth boundary Γ. In this section we will study the generalized Buckley-Leverett model (1), (5) for given $\nu, \tau > 0$:

$$\begin{aligned} \partial_t u + \operatorname{div}\big(\mathbf{v}\ g(u)\big) &= 0 \qquad \operatorname{div}(\mathbf{v}) = 0, && (6)\\ \tau\,\partial_t \mathbf{v} - \nu\Delta \mathbf{v} + h(u)\mathbf{v} &= -\nabla p, \qquad \text{in } \Omega_T := \Omega \times (0, T), && (7) \end{aligned}$$

satisfying the boundary-initial conditions

$$(u, \mathbf{v}) = (u_b, \mathbf{b}) \quad \text{on } \Gamma_T := \Gamma \times (0, T) \quad \text{and} \quad (u, \mathbf{v})|_{t=0} = (u_0, \mathbf{v}_0) \quad \text{in } \Omega. \tag{8}$$

Before the formulation of the main result let us introduce the following spaces

$$\begin{aligned} \mathbf{V}^s(\Omega) \ &: = \{\mathbf{u} \in H^s(\Omega) : \quad \operatorname{div}(\mathbf{u}) = 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\, d\mathbf{x} = 0\},\\ \mathbf{V}^s(\Gamma) \ &: = \{\mathbf{u} \in H^s(\Gamma) : \quad \int_\Gamma \mathbf{u}\cdot\mathbf{n}\, d\mathbf{x} = 0\}, \qquad \mathbf{V}^{-s}(\Gamma) := (\mathbf{V}^s(\Gamma))',\\ \mathbf{G}(\Gamma_T) \ &: = \left\{\mathbf{u} \in L^2(0, T; \mathbf{V}^{1/2}(\Gamma)) : \quad \partial_t \mathbf{u} \in L^2(0, T; \mathbf{V}^{-1/2}(\Gamma)\right\}, \end{aligned}$$

where $\mathbf{n} = \mathbf{n}(\mathbf{x})$ is the outside normal to Ω at $\mathbf{x} \in \Gamma$. We assume that our data satisfy the following regularity properties

$$\begin{aligned} g,\ h \ &\in\ W^{1,\infty}_{\text{loc}}(\mathbb{R}) \qquad \text{with} \quad 0 < h_0 \le h(u),\\ 0 \ &\leqslant\ u_b \leqslant 1 \quad \text{on } \Gamma_T, \qquad 0 \leqslant u_0 \leqslant 1 \quad \text{on } \Omega, && (9)\\ \mathbf{v}_0 \ &\in\ \mathbf{V}^0(\Omega), \qquad \mathbf{b} \in \mathbf{G}(\Gamma_T) \qquad \text{and}\\ \mathbf{b}(0)\cdot\mathbf{n} \ &=\ \mathbf{v}_0\cdot\mathbf{n} \qquad \text{in} \quad H^{-1/2}(\Gamma). && (10) \end{aligned}$$

Now, since equation (6) is a hyperbolic scalar conservation law, the saturation function u may admit shocks. Therefore, in order to select a correct physical solution, we need the entropy concept of solutions [8], as given by the following

Definition 2.1. *A pair of functions* $u \in L^\infty(\Omega_T)$, $\mathbf{v} \in L^2(0,T;\mathbf{V}^1(\Omega))$ *is called a weak solution of system* (6)-(8), *if this pair satisfies:*

1) the integral inequality

$$\iint_{\Omega_T} \big(|u-v|\ \phi_t + \operatorname{sgn}(u-v)\big(g(u)-g(v)\big)\ \mathbf{v}\cdot\nabla\phi\big)\ d\mathbf{x}\,dt$$
$$+ \mathrm{K}\int_{\Gamma_T} |\mathbf{b}\cdot\mathbf{n}|\ |u_b - v|\phi\ d\mathbf{x}\,dt + \int_\Omega |u_0 - v|\phi(0,x)\ d\mathbf{x} \geqslant 0 \tag{11}$$

for $\forall v \in \mathbb{R}$ *and for any positive* $\phi \in C_0^\infty((-\infty,T)\times\mathbb{R}^d)$. *Here* $\mathrm{K}{:=}||g'||_{L^\infty(\mathbb{R})}$.

2) equation (8) *in the usual distributional sense and* $\mathbf{v}|_{\Gamma_T} = \mathbf{b}$.

Theorem 2.2. *If the data* g, h, u_b, u_0, $\mathbf{v}_0$, $\mathbf{b}$ *have the regularity* (9)-(10), *then system* (6)-(8) *has a weak solution, such that* $\mathbf{v} \in H^1(0,T;\mathbf{V}^{-1}(\Omega))$,

$$0 \leqslant u \leqslant 1 \qquad a.e.\ in \quad \Omega_T,$$
$$\sqrt{\tau}\|\mathbf{v}\|_{C([0,T];\mathbf{V}^0(\Omega))} + \|\mathbf{v}\|_{L^2(0,T;\mathbf{V}^1(\Omega))} + \tau\|\mathbf{v}\|_{H^1(0,T;\mathbf{V}^{-1}(\Omega))} \leqslant C, \tag{12}$$

where C *is a positive constant independent of* τ.

The generalized Buckley-Leverett model (6)-(8) has **specific difficulties** compared to *the usual theory of quasi-linear scalar conservation laws*:

1) It is not possible to obtain ANY a priori compactness for approximate solutions (no BV-bounds or L_1- Kruzkov's continuous compactness);

2) Since we are leading with the initial-boundary problem in the class of $L_\infty-$bounded solutions, such solutions do not have the traces values of Sobolev. Moreover this difficulty is also related with a so-called *boundary layer problem.*

To overcome these 2 difficulties, we use the Kinetic Theory [9] and a concept of trace values for irregular functions, developed for divergence type equations (see, for instance [10]).

2.1. Approximated system

In order to show the solvability of system (6)-(8), first we study the following approximated parabolic system for a fixed $\varepsilon > 0$

$$\partial_t u^\varepsilon + \operatorname{div}\big(\mathbf{v}^\varepsilon\ g(u^\varepsilon)\big) = \varepsilon\ \Delta u^\varepsilon \qquad \text{in } \Omega_T, \tag{13}$$
$$\tau\,\partial_t \mathbf{v}^\varepsilon - \nu\Delta\mathbf{v}^\varepsilon + h(u^\varepsilon)\mathbf{v}^\varepsilon = -\nabla p^\varepsilon, \quad \operatorname{div}(\mathbf{v}^\varepsilon) = 0 \quad \text{in } \Omega_T \tag{14}$$

jointly with the boundary-initial conditions

$$\varepsilon\frac{\partial u^\varepsilon}{\partial \mathbf{n}} + M(u^\varepsilon - u_b^\varepsilon) = 0 \qquad \text{and} \qquad \mathbf{v}^\varepsilon = \mathbf{b} \quad \text{on} \quad \Gamma_T$$
$$(u^\varepsilon, \mathbf{v}^\varepsilon)|_{t=0} = (u_0^\varepsilon, \mathbf{v}_0) \quad \text{in } \Omega, \tag{15}$$

where u_b^ε, u_0^ε are regularized boundary-initial data satisfying suitable compatibility conditions. Using the results of parabolic-elliptic theory, we can get the solvability of system (13)–(15).

Proposition 2.3. *For each $\varepsilon > 0$, there exists a unique solution $(u^\varepsilon, \mathbf{v}^\varepsilon)$ of system (13)–(15), which has the following regularity $u^\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$ and $\mathbf{v}^\varepsilon \in L^2(0,T;\mathbf{V}^1(\Omega)) \cap H^1(0,T;\mathbf{V}^{-1}(\Omega))$ satisfying*

$$
\begin{aligned}
0 \leqslant u^\varepsilon \leqslant 1 \quad a.e.\ on\ \Omega_T, \qquad \varepsilon\|\nabla u^\varepsilon\|^2_{L^2(\Omega_T)} &\leqslant C && (16)\\
\sqrt{\tau}\|\mathbf{v}^\varepsilon\|_{C([0,T];\mathbf{V}^0(\Omega))} + \|\mathbf{v}^\varepsilon\|_{L^2(0,T;\mathbf{V}^1(\Omega))} + \tau\|\mathbf{v}^\varepsilon\|_{H^1(0,T;\mathbf{V}^{-1}(\Omega))} &\leqslant C, && (17)
\end{aligned}
$$

where C is a positive constant independent of ε (and τ).

2.2. Kinetic approach. Sketch of the proof of Theorem 2.2

Let $(\eta(u), q(u))$ be an entropy pair for equation (6), i.e. $\eta = \eta(u)$ is a Lipschitz continuous convex function and $q'(u) = \eta'(u)g'(u)$ for $u \in \mathbb{R}$. Then from (13), we have in the distributional sense

$$\partial_t\eta(u^\varepsilon) + \operatorname{div}(\mathbf{v}^\varepsilon q(u^\varepsilon)) - \varepsilon\,\Delta\eta(u^\varepsilon) = -\varepsilon\,\eta''(u)\,|\nabla\eta(u)|^2 \leqslant 0. \tag{18}$$

For instance, we can take the entropy pair defined by

$$\eta(u) := |u-v|^+, \qquad q(u) := \operatorname{sgn}^+(u-v)\,(g(u)-g(v)) \quad \text{for each } v \in \mathbb{R}.$$

Then, we have in the distributional sense

$$\partial_t|u^\varepsilon - v|^+ + \operatorname{div}\left[\mathbf{v}^\varepsilon \operatorname{sgn}^+(u^\varepsilon - v)\,(g(u^\varepsilon) - g(v))\right] - \varepsilon\,\Delta|u^\varepsilon - v|^+ = -m^\varepsilon. \tag{19}$$

Here $|v|^+ := \max\{v,\ 0\}$, $\operatorname{sgn}^+(v) := 1$, if $v > 0$; 0, if $v \leqslant 0$ and m^ε is a real non-negative Radon measure. The differentiation of (19) on the variable v gives that the function $f^\varepsilon(t,\mathbf{x},v) := \operatorname{sgn}^+(u^\varepsilon(t,\mathbf{x}) - v)$ satisfies

$$\partial_t f^\varepsilon + g'(v)\,\mathbf{v}^\varepsilon \cdot \nabla f^\varepsilon - \varepsilon\,\Delta f^\varepsilon = \partial_v m^\varepsilon \qquad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \tag{20}$$

Let us point out that $0 \leqslant f^\varepsilon(t,\mathbf{x},v) \leqslant 1$ in $\Omega_T \times \mathbb{R}$. It is possible to show that m^ε is uniformly bounded with respect to ε. Hence due to Proposition 2.3, there exist subsequences of f^ε, $\mathbf{v}^\varepsilon$, m^ε and the functions

$$f \in L^\infty(\Omega_T \times \mathbb{R}), \quad \mathbf{v} \in L^2(0,T;\mathbf{V}^1(\Omega)) \tag{21}$$

and a real nonnegative Radon measure $m = m(t,\mathbf{x},v)$, such that

$$
\begin{aligned}
f^\varepsilon &\to f && \star\text{-weakly in } L^\infty(\Omega_T \times \mathbb{R}),\\
\mathbf{v}^\varepsilon &\to \mathbf{v}, && \varepsilon\,\nabla u^\varepsilon \to 0 \quad \text{strongly in } L^2(\Omega_T),\\
m^\varepsilon &\to m && \text{weakly in } \mathcal{M}^+_{loc}(\overline{\Omega}_T \times \mathbb{R}),
\end{aligned}
$$

Since (20) is linear, it follows that

$$\partial_t f + g'(v)\,\mathbf{v} \cdot \nabla f = \partial_v m \qquad \text{in } \mathcal{D}'(\Omega_T \times \mathbb{R}). \tag{22}$$

The non-regular function $f \in L^\infty(\Omega_T \times \mathbb{R})$ can not have trace values in the usual sense of Sobolev, that brings one of the major difficulty for a studied problem. Fortunately, using the divergence type form of equation (22), it is possible to introduce a concept of the trace values for f (see some theoretical explanation of it in the article [10]). Accounting the initial-boundary conditions for f^ε, we can show that

$$\begin{aligned} f &= \operatorname{sgn}^+(u_0 - v) \qquad \text{for } t = 0 \\ f &= \operatorname{sgn}^+(u_b - v) \qquad \text{on} \quad \Gamma_T \times \mathbb{R}, \quad \text{where } g'(v)\mathbf{b}\cdot\mathbf{n} < 0. \end{aligned} \tag{23}$$

Due to $\mathbf{v} \in L^2(0, T; \mathbf{V}^1(\Omega))$, we can apply DiPerna-Lions's theory for transport equations [11], and deduce that the solution f of (22)-(23) takes values equals only to 0 and 1 in $\Omega_T \times \mathbb{R}$. Since $f(\cdot, \cdot, v)$ is a monotone function on v, as a limit of monotone functions $f^\varepsilon(\cdot, \cdot, v)$, there exists a function $z = z(t, \mathbf{x})$:

$$f = \operatorname{sgn}^+(z(t, \mathbf{x}) - v).$$

Therefore we derive the *-weak convergence in $L^\infty(\Omega_T)$

$$G(u^\varepsilon) = \int_0^1 G'(v) f^\varepsilon(\cdot, \cdot, v)\, dv \rightharpoonup \int_0^1 G'(v) f(\cdot, \cdot, v)\, dv = G(z)$$

for any $G \in C^1([0, 1])$, $G(0) = 0$. This implies $z = u$ and

$$u^\varepsilon \to u \qquad \text{strongly in } L^p(\Omega_T) \quad \text{for } \forall p < \infty.$$

Hence the function $\mathbf{v}$ fulfills equality (7). And moreover if we take Kružkov's entropy pair

$$\eta(u) := |u - v|, \qquad q(u) := \operatorname{sgn}(u - v)\,(g(u) - g(v)) \quad \text{for } \forall v \in \mathbb{R}. \tag{24}$$

in inequality (18) and pass to the limit on $\varepsilon \to 0$, we deduce that u satisfies (11), that ends the proof of Theorem 2.2.

3. Generalized Buckley-Leverett model (1), (5)

In this section we formulate a solvability result for the generalized Buckley-Leverett model (1), (5) with a given viscous parameter $\nu > 0$:

$$\begin{aligned} \partial_t u + \operatorname{div}\big(\mathbf{v}\, g(u)\big) &= 0 \qquad \operatorname{div}(\mathbf{v}) = 0, \\ -\nu\Delta\mathbf{v} + h(u)\mathbf{v} &= -\nabla p, \qquad \text{in } \Omega_T \end{aligned} \tag{25}$$

satisfying the boundary-initial conditions

$$(u, \mathbf{v}) = (u_b, \mathbf{b}) \quad \text{on } \Gamma_T \quad \text{and} \quad u = u_0 \quad \text{in } \Omega. \tag{26}$$

Theorem 3.1. *Let the data g, h, u_b, u_0, $\mathbf{b}$ satisfy the regularity properties* (9) *and* $\mathbf{b} \in \mathbf{G}(\Gamma_T)$. *Then system* (25)-(26) *has a weak solution* $(u, \mathbf{v})$, *which is understood in the sense of Definition* (2.1) *for* $\tau = 0$, *such that*

$$\begin{aligned} 0 &\leqslant u \leqslant 1 \qquad a.\ e.\ in \quad \Omega_T, \\ \mathbf{v},\ \partial_t\mathbf{v} &\in L^2(0, T; \mathbf{V}^1(\Omega)). \end{aligned}$$

To show the above theorem, we can use Theorem 2.2. Due to the last mentioned theorem system (6)-(8) admits a solution $(u^\tau, \mathbf{v}^\tau)$, satisfying (12) for a fixed $\tau > 0$. Hereupon the issue is to pass to the limit on $\tau \to 0$. Of course, estimates (12) are not sufficient for the limit transition on $\tau \to 0$ in system (6)-(8), since we need the strong convergence of a subsequence for $\{u^\tau\}_{\tau>0}$. To get this strong convergence, we can apply the Kinetic approach, developed in Section 2.2 and prove Theorem 3.1.

Finally let us observe that, the interesting reader can find a more full information concerning developed technics applied in Theorem 2.2, Theorem 3.1 in the articles [12]-[14].

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