BOYCOTT EFFECT IN TWO-DIMENSIONAL SEDIMENTATION WITH DIFFUSION

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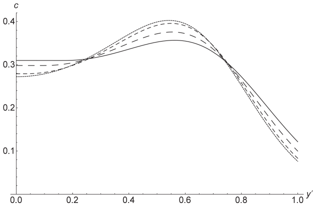
Different approaches:

- **Empirical correlations** (calculation in seconds)
  - One velocity model of mixture
- **Algebraic slip model** (minutes)
  - Solve one momentum equation for the mixture
- **Two-fluids theory** (minutes, hours)
  - Solve as many momentum equations as there are phases
- **Discrete element methods** (hours, days)
- ...

Our main goal is to develop mathematical model of two-phase mixture which will be thermodynamically correct in general case and to perform some test calculations using this model.

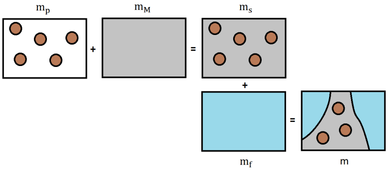
The method we used was proposed by Landau and Khalatnikov for superfluid ^{2}He.

- In one-velocity case such method was applied for micropolar fluid in Shelukhin, Neverov 2016.
- More general case for two-velocity model was considered in Shelukhin, 2018.



Both models agreed with Segre-Silberberg effect (Segre-Silberberg, Nature, 1961)

Conservation laws without dissipation



Variables:

$$\rho = \frac{m_p + m_M + m_f}{V}, \quad \rho_p = \frac{m_p}{V},$$

$$\rho_s = \frac{m_p + m_M}{V}, \quad \rho_f = \frac{m_f}{V},$$

$$\phi_i = \frac{V_i}{V}, \quad c = \frac{m_p}{m}, \quad \overline{\rho_i} = \frac{m_i}{V_i}$$

where V is the unit volume, m_p — mass of particles in volume V, m_M — mass of mud in volume V, m_f — mass of fluid in volume V, ρ_i — partial densities, $\overline{\rho_i}$ — true densities, ϕ_k — volume concentration of particles, c — mass concentration of particles and $i = f, s$.

It follows from the definition that $\rho_i = \overline{\rho_i}\phi_i$.

Also we call E_0 internal energy of unit volume, $S = \rho\eta$ where η is specific entropy, θ — temperature, μ — chemical potential (V is constant)

$$dE_0 = \theta dS + \mu d\rho + \mu_p d(\rho c) + \mathbf{u} d\mathbf{j}_b, \quad \mathbf{j}_b = \rho_s \mathbf{u}, \quad \mathbf{u} = \mathbf{v}_s - \mathbf{v}_f. \quad (1)$$

We assume that two-phase granular fluids without dissipation obey the following equations:

$$\rho_t + \mathrm{div}\mathbf{j} = 0, \quad \mathbf{j}_t + \mathrm{div}\Pi = 0, \quad S_t + \mathrm{div}\mathbf{F}_\theta = 0, \quad (2)$$

$$E_t + \mathrm{div}\mathbf{Q} = 0, \quad (\rho_s)_t + \mathrm{div}\,(\rho_s \mathbf{v}_s) = 0, \quad (\rho c)_t + \mathrm{div}\mathbf{F}_c = 0, \quad (3)$$

$$(\mathbf{v}_f)_t + \frac{\partial \mathbf{v}_f}{\partial \mathbf{x}} \langle \mathbf{v}_f \rangle = \alpha \nabla \mu + \beta \nabla \theta + \gamma \nabla \mu_p \quad (4)$$

with unknown fluxes $\mathbf{j}$, Π, $\mathbf{F}_\theta$, $\mathbf{Q}$, $\mathbf{F}_c$.

Galilean transformations

The quantities $\mathbf{j}_b$, Π_b, E_b, and $\mathbf{Q}_b$ assigned to fluid frame of reference are related to the laboratory frame of reference quantities $\mathbf{j}$, Π, E, and $\mathbf{Q}$ by the following Galilean transformations

$$E = E_b + \rho \frac{\mathbf{v}_f}{2} + \mathbf{v}_f \cdot \mathbf{j}_b, \quad \mathbf{j}_b = \rho_s \mathbf{u}, \quad \mathbf{u} = \mathbf{v}_s - \mathbf{v}_f, \tag{5}$$

$$\mathbf{j} = \rho \mathbf{v}_s + \mathbf{j}_b, \quad \Pi = \Pi_b + + \rho \mathbf{v}_f \otimes \mathbf{v}_f + \mathbf{v}_f \otimes \mathbf{j}_b + \mathbf{j}_b \otimes \mathbf{v}_f, \tag{6}$$

$$\mathbf{Q} = \mathbf{Q}_b + \left(\rho \frac{\mathbf{v}_f^2}{2} + \mathbf{j}_b \cdot \mathbf{v}_f + E_b \right) \mathbf{v}_f + \mathbf{j}_b \frac{\mathbf{v}_f^2}{2} + \Pi_b \langle \mathbf{v}_f \rangle. \tag{7}$$

System (2) – (7) is overdetermined, so we can identify unknowns treating the energy conservation law as a consequence of other equations.

Two-fluid model without dissipation

The model for two-phase granular fluid without dissipation have been obtained

$$\frac{\partial(\rho_s \mathbf{v}_s)}{\partial t} + \text{div}\,(\rho_s \mathbf{v}_s \otimes \mathbf{v}_s) = -\frac{\rho_s}{\rho}\nabla p - \frac{\rho_s \rho_f}{2\rho}\nabla \mathbf{u}^2, \quad \mathbf{u} = \mathbf{v}_s - \mathbf{v}_f,$$

$$\frac{\partial(\rho_f \mathbf{v}_f)}{\partial t} + \text{div}\,(\rho_f \mathbf{v}_f \otimes \mathbf{v}_f) = -\frac{\rho_f}{\rho}\nabla p + \frac{\rho_s \rho_f}{2\rho}\nabla \mathbf{u}^2,$$

$$\frac{\partial(\rho c)}{\partial t} + \text{div}\,(c\mathbf{j}) = 0, \quad \mathbf{j} = \rho_s \mathbf{v}_s + \rho_f \mathbf{v}_f, \quad \rho = \rho_s + \rho_f,$$

$$S_t + \text{div}\frac{S\mathbf{j}}{\rho} = 0, \quad \rho_{st} + \text{div}\,(\rho_s \mathbf{v}_s) = 0, \quad \rho_{ft} + \text{div}\,(\rho_f \mathbf{v}_f) = 0,$$

Now we pay attention to irreversible processes with dissipation.

So, we look for conservation laws

$$\rho_t + \operatorname{div}\mathbf{j} = 0, \quad \mathbf{j}_t + \operatorname{div}(\Pi + \pi) = 0, \quad S_t + \operatorname{div}\left(\mathbf{F}_\theta + \frac{\mathbf{q}}{\theta}\right) = \frac{R}{\theta}, \tag{8}$$

$$\mathbf{v}_{ft} + \nabla \mathbf{v}_f \langle \mathbf{v}_f \rangle = -\frac{1}{\rho}\left(\nabla p - \frac{\rho_1}{2}\nabla \mathbf{u}^2\right) + \mathbf{f}_2, \tag{9}$$

$$(\rho c)_t + \operatorname{div}(\mathbf{F}_c + \mathbf{l}) = 0, \quad E_t + \operatorname{div}(\mathbf{Q} + \mathbf{Q}_1) = 0, \tag{10}$$

with unknowns $\xi = \{\pi, \mathbf{l}, \mathbf{q}, \mathbf{f}_2, \mathbf{Q}_1, R\}$. Here, R is the entropy production and $\mathbf{q}$ is the heat flux.

Taking into account that system (8) – (10) is overdetermined and $R \geq 0$ we identify the unknown fluxes $\pi, \mathbf{l}, \mathbf{q}, \mathbf{f}_2, \mathbf{Q}_1, R$.

Mathematical model for two-phase mixture with dissipation:

$$\frac{\partial(\rho_s \mathbf{v}_s)}{\partial t} + \text{div}\,(\rho_s \mathbf{v}_s \otimes \mathbf{v}_s) = -\frac{\rho_s}{\rho}\nabla p - \frac{\rho_s \rho_f}{2\rho}\nabla \mathbf{u}^2 - k\,\mathbf{u} + \text{div}\,T_s + \rho_s \mathbf{g}, \tag{11}$$

$$\frac{\partial(\rho_f \mathbf{v}_f)}{\partial t} + \text{div}\,(\rho_f \mathbf{v}_f \otimes \mathbf{v}_f) = -\frac{\rho_f}{\rho}\nabla p + \frac{\rho_s \rho_f}{2\rho}\nabla \mathbf{u}^2 + k\,\mathbf{u} + \text{div}\,T_f + \rho_f \mathbf{g}, \tag{12}$$

$$\frac{\partial(\rho c)}{\partial t} + \text{div}\,(c\mathbf{j} + \mathbf{l}) = 0, \quad \mathbf{j} = \rho_s \mathbf{v}_s + \rho_f \mathbf{v}_f, \quad \rho = \rho_s + \rho_f, \tag{13}$$

$$S_t + \text{div}\left(\frac{S\mathbf{j}}{\rho} + \frac{\mathbf{q}}{\theta}\right) = \frac{R}{\theta}, \quad \mathbf{u} = \mathbf{v}_s - \mathbf{v}_f, \tag{14}$$

$$\rho_{st} + \text{div}\,(\rho_s \mathbf{v}_s) = 0, \quad \rho_{ft} + \text{div}\,(\rho_f \mathbf{v}_f) = 0. \tag{15}$$

Two-fluid model with dissipation

Generalized Fick law for diffusion flux:

$$\mathbf{l} = \gamma_1 \nabla p + \gamma_2 \nabla \theta + \gamma_3 \nabla c + \gamma_4 \nabla \mathbf{u}^2 + \rho c B \mathbf{g}. \tag{16}$$

Constitutive equations:

$$T_i = 2\eta_i D_i' + 2\lambda_i \mathrm{div}\mathbf{v}_i \cdot \mathrm{I}, \quad 2D_i = \nabla \mathbf{v}_i + (\nabla \mathbf{v}_i)^* \tag{17}$$

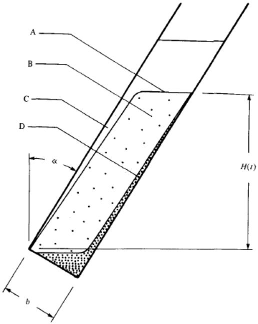
where D' is the deviatoric part of D:

$$D' = D - \frac{1}{3}\mathrm{div}\mathbf{v} \cdot \mathrm{I}.$$

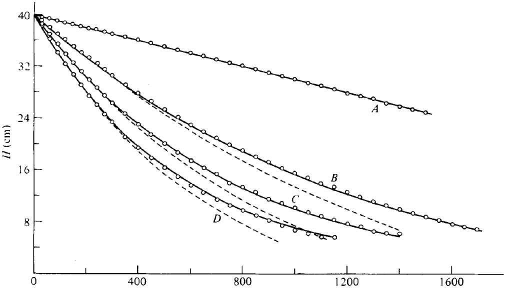
Flux $\mathbf{l}$ should be zero if $c = 0$ or $c = 1$, consequently

$$\gamma_i = c\,(1 - c)\,\gamma_i^0$$

Boycott (Nature, 1920): "... if oxalated or defibrinated blood is put to stand in narrow tubes, the corpuscles sediment a good deal faster if the tube is inclined than when it is vertical."



a) Scheme



b) Experimental results

Figure 1: Experimental results from Acrivos et. al. (1979), a) — inclined cell and regions in the flow field, b) — height of the top interface $H(t)$ for $c_0 = 0.1$ and for: A, $\alpha = 0°$; B, $\alpha = 20°$; C, $\alpha = 35°$; D, $\alpha = 50°$.

Boycott effect

Some ways to estimate Boycott effect:

- PNK theory (Ponder, Nakamura and Kuroda) — geometrical
  Surface area available for sedimentation increases with inclination angle α.

$$S(t) = \frac{v_0 b}{\cos\alpha}\left(1 + \frac{H}{b}\sin\alpha\right),$$

  where $S(t)$ — volumetric rate at which clarified fluid is formed.

- Two-fluid hydrodynamic models with nonstationary forces:
  - Yu. A. Nevskii, A. N. Osiptsov, Slow Gravitational Convection of Disperse Systems in Domains with Inclined Boundaries, 2010, Fluid Dynamics;
  - Sergio Palma, Christian Ihle et. al., Particle Organization After Viscous Sedimentation in Tilted Containers, 2016, Physics of Fluids;
- The lattice Boltzmann method
  - D.M. Snider, An Incompressible Three-Dimensional Multiphase Particle-in-Cell Model for Dense Particle Flows, 2001, Journal of Computational Physics;
  - Zu-Jia Xu et. al., A Numerical Simulation of the Boycott Effect, 2005, Chem. Eng. Comm.;

2D sedimentation

Let us consider 2D sedimentation in the domain Ω

$$\Omega = \{0 < x < a_1, \quad 0 < y < a_2\}$$

We pass to dimensionless variables and introduce parameters

$$\mathrm{Re} = \frac{a_1 V \rho_f}{\eta_f}, \quad k_1 = \frac{k a_1}{\rho V}, \quad g_1 = \frac{g a_1}{V^2}, \quad \mathbf{g}_1 = -g_1 \mathbf{e}_y,$$

$$\Gamma_1 = \frac{\gamma_1 \rho V^2}{a_1 l_0}, \quad \Gamma_3 = \frac{\gamma_3}{a_1 l_0}, \quad \Gamma_4 = \frac{\gamma_4 V^2}{a_1 l_0}, \quad \Gamma_5 = \frac{\rho c B V^2}{a_1 l_0}.$$

Let us formulate boundary and initial conditions:

$$\partial\Omega: \qquad \mathbf{v}_s = 0, \quad \mathbf{v}_f = 0, \quad \nabla c \cdot \mathbf{n} = 0, \quad \nabla p \cdot \mathbf{n} = 0, \tag{18}$$

$$t = 0: \qquad \mathbf{v}_s = 0, \quad \mathbf{v}_f = 0, \quad c = c_0. \tag{19}$$

We suppose that the whole mixture is incompressible $\rho = \mathrm{const}$ to simplify model.

$$\frac{\partial \mathbf{v}_s}{\partial t} + \nabla \mathbf{v}_s \langle \mathbf{v}_s \rangle = -\nabla p + \frac{1}{\mathrm{Re}} \frac{\rho_f}{\rho_s} \operatorname{div} \left(\frac{2\eta_s}{\eta_f} D_s \right) - k_1 \frac{\rho}{\rho_s} \mathbf{u} - \frac{\rho_f}{2\rho} \nabla \mathbf{u}^2 + \mathbf{g}_1, \tag{20}$$

$$\frac{\partial \mathbf{v}_f}{\partial t} + \nabla \mathbf{v}_f \langle \mathbf{v}_f \rangle = -\nabla p + \frac{1}{\mathrm{Re}} \operatorname{div} (2D_f) + k_1 \frac{\rho}{\rho_f} \mathbf{u} + \frac{\rho_s}{2\rho} \nabla \mathbf{u}^2 + \mathbf{g}_1, \tag{21}$$

$$\operatorname{div} \mathbf{v} = 0, \quad \mathbf{v} \equiv \frac{\rho_s}{\rho} \mathbf{v}_s + \frac{\rho_f}{\rho} \mathbf{v}_f, \tag{22}$$

$$\frac{\partial c}{\partial t} + \operatorname{div} (c\mathbf{v} + \mathbf{l}) = 0, \quad \mathbf{l} = -\left(\Gamma_3 \nabla c + \Gamma_1 \nabla p + \Gamma_4 \nabla \mathbf{u}^2 \right) + \Gamma_5 \mathbf{g}_1. \tag{23}$$

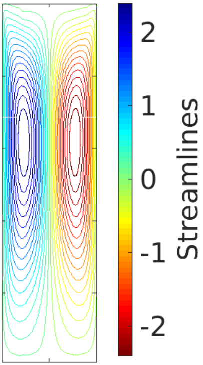
Problem was solved via open source package FreeFEM++ using the projection algorithm for velocity-pressure equations.

Numerical results: streamlines

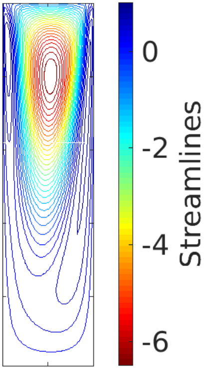
Figure depicts streamlines of total velocity in vertical and inclined cell

In the vertical case one can see two opposite-sign vortexes (a) but in inclined cell only one vortex appears (b) and mixture moves faster.

Such flow pattern was described earlier (Guazelli 2006, Snider 2001)



a) vertical



b) inclined

The figure 2 depicts a comparison between concentration distribution in vertical and inclined cell for different times.

Mass concentration c is contained in interval $[0, 1]$ for any time

As one can see dispersed phase accumulates faster in inclined cell.

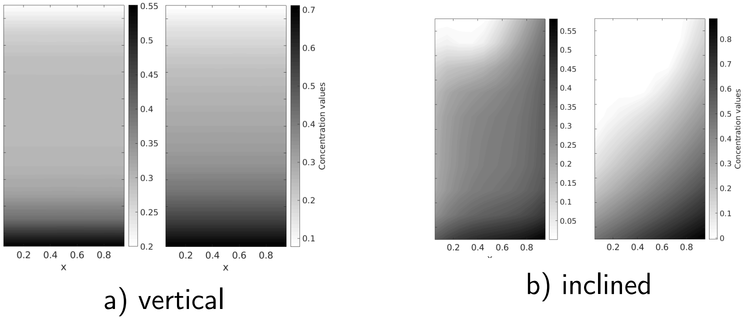


Figure 2: mass concentration for vertical (a) and inclined (b) cell at dimensionless time $t' = 1$ (left) and $t' = 3$ (right)

Generalized height of clear fluid interface

$$H(t) = \frac{\frac{\int \xi d\Omega}{\Omega}}{a_1}, \quad \text{where } \xi = \begin{cases} 1, & \text{if } c < 10^{-4} \\ 0, & \text{if } c \geq 10^{-4} \end{cases}$$

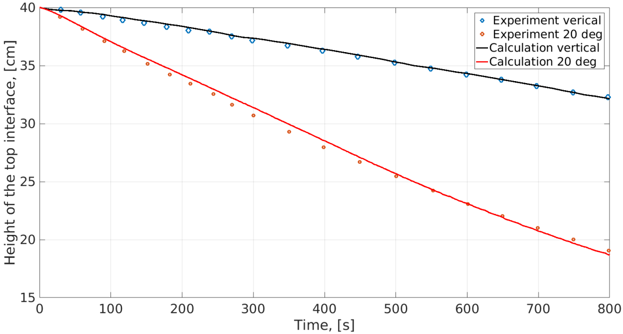


Figure 3: Height of the top clear fluid interface. Comparison between experiment and calculations for vertical and inclined cell.

Conclusions

- Two-phase model which is in accordance with thermodynamic principles was obtained
- The equations derived are applied to 2D problem of particle sedimentation
- Boycott effect was confirmed and it was showed numerically that mass concentration $c \in [0, 1]$ for any time
- A comparison between the experiments and a numerical simulation was performed