On the identification of the piezo-conductivity coefficient in the pseudoparabolic equation of filtration type

A.Sh. Lyubanova
Siberian Federal University, Russia
e-mail: `lubanova@mail.ru`

The report discusses the inverse problem on determination of an unknown coefficient in the second order term of the multi-dimensional linear pseudoparabolic equation of the third order under the initial data and the Dirichlet boundary condition. The integral condition of overdetermination on the boundary is taken as additional data for the identification of the unknown coefficient. The assumptions on the input data are formulated wherein the local existence and uniqueness of the solution of the inverse problem is proved.

1. Introduction. The statement of the problem and preliminaries

An inverse problem for the pseudoparabolic equation

$$(u + L_1 u)_t + L_2 u = f \tag{1.1}$$

with the differential operators L_1 and L_2 of the second order in spacial variables is discussed in this paper. Applications of the problem deal with the recovery of unknown parameters indicating physical properties of a medium (see [1], [2], [3]). Since the natural stratum is involved, the parameters in (1.1) should be determined on the basis of the investigation of its behaviour under the natural non-steady-state conditions. This leads to the interest in studying the inverse problems for (1.1) and its analogue.

A variety of works are devoted to the inverse problems for (1.1) (see [4], [5], [6] and references therein). The results of [4],[6] are concerned with the reconstructing of unknown source f and the kernels in integral term of (1.1) with the integro-differential operator L_2. To the present author's knowledge, inverse problems of the identification of unknown variable coefficients in the terms of the second and third order of (1.1) have not been studied yet.

Let Ω be a domain in $\mathbf{R}^n$ with a boundary $\partial\Omega \in C^2$, T an arbitrary real number and $Q_T = \Omega \times (0, T)$. Throughout this paper we use the notation $\|\cdot\|$ and $(\cdot,\cdot)$ for the norm and the inner product of $L^2(\Omega)$; $\|\cdot\|_j$ and $\langle \cdot,\cdot \rangle_j$ are the norm of $W_2^j(\Omega)$ and the duality relation between $\overset{\circ}{W}{}_2^j(\Omega)$ and $W_2^{-j}(\Omega)$, respectively $(j = 1, 2)$; as usual $W_2^0(\Omega) = L^2(\Omega)$.

Let us introduce linear differential operators $M : W_2^1(\Omega) \to (W_2^1(\Omega))^*$ and $L : W_2^1(\Omega) \to L^2(\Omega)$ of the form

$$Mv = -\mathrm{div}(\mathbf{M}(x)\nabla v) + m(x)v, \qquad Lv = \sum_{i=1}^{n} l_i(x) v_{x_i} + l(x)v$$

where $\mathbf{M}(x) \equiv (m_{ij}(x))$ is a matrix of functions $m_{ij}(x),\ i, j = 1, 2, \dots, n$.

The aim of the paper is to establish the local existence and uniqueness of the solution to the inverse problem of finding the coefficient $k(t)$ in (1.1) with $L_1 = M$, $L_2 = k(t)M + L$ given the additional boundary data.

We assume that the following conditions are fulfilled.

I. $m_{ij}(x)$, $\partial m_{ij}/\partial x_r$, $i, j, r = 1, 2, \ldots, n$, and $m(x)$ are bounded in Ω. M is an operator of elliptic type, that is, there exist positive constants m_1 and m_2 such that for any $v \in \overset{\circ}{W}{}_2^1(\Omega)$

$$m_1\|v\|_1^2 \leq \langle Mv, v\rangle_1 \leq m_2\|v\|_1^2. \tag{1.2}$$

II. There exists a positive constant m_3 such that for any $v \in W_2^2(\Omega)$

$$\|Mv\| \leq m_3\|v\|_2. \tag{1.3}$$

III. $m_{ij}(x) = m_{ji}(x)$ for $i, j = 1, 2, \ldots, n$ and $m(x) \geq 0$ for $x \in \Omega$.

IV. $l_i(x)$, $i, j, l = 1, 2, \ldots, n$, and $l(x)$ are bounded in Ω. For any $v \in W_2^1(\Omega)$

$$\|Lv\|_0 \ \leq \ \lambda\|v\|_1 \tag{1.4}$$

with a positive constant λ.

We are studying the following inverse problem.

Problem 1. For a given constant η and functions $f(t, x)$, $g(t, x)$, $\beta(t, x)$, $U_0(x)$, $\omega(t, x)$, $\varphi_1(t)$, $\varphi_2(t)$ find the pair of functions $(u(t, x), k(t))$ satisfying the equation

$$u_t + \eta M u_t + k(t)\, Mu + Lu = f(t, x), \quad (t, x) \in Q_T,$$

and the conditions

$$(u + \eta Mu)\big|_{t=0} = U_0(x), \quad x \in \Omega,$$

$$u\big|_{\partial\Omega} = \beta(t, x), \quad t \in [0, T],$$

$$\int_{\partial\Omega} \left\{\eta\frac{\partial u_t}{\partial \nu} + k(t)\frac{\partial u}{\partial \nu}\right\} \omega(t, x)\, \mathrm{d}S + \varphi_1(t)k(t) = \varphi_2(t), \quad t \in (0, T). \tag{1.5}$$

Here $\frac{\partial}{\partial \nu} = (\mathbf{n}, \mathbf{M}(x)\nabla)$ and $\mathbf{n}$ is the unit outward normal to $\partial\Omega$.

If $\omega(t, x) \equiv 1$ and $\varphi_1 \equiv 0$, then the integral condition of overdetermination (1.5) means a given flux of a liquid through the surface $\partial\Omega$, for instance, the total discharge of a liquid through the surface of the ground. Similar nonlocal conditions were considered in [7],[8].

We introduce functions $a(t, x)$, $h^\eta(t, x)$, $b(t, x)$ and $b^\eta(t, x)$ as the solutions of the Dirichlet problems

$$Ma = 0 \quad b^\eta + \eta M b^\eta = 0 \quad \text{in } \Omega, \quad \beta(t, x), a\big|_{\partial\Omega} = b^\eta\big|_{\partial\Omega} = \beta(t, x);$$

$$Mb = 0 \quad h^\eta + \eta M h^\eta = 0 \quad \text{in } \Omega, \quad b\big|_{\partial\Omega} = h^\eta\big|_{\partial\Omega} = \omega(t, x), \tag{1.6}$$

respectively, and keep the following notation:

$$\langle Mv_1, v_2\rangle_{1,M} = (\mathcal{M}(x)\nabla v_1, \nabla v_2) + (m(x)v_1, v_2), \quad v_1, v_2 \in W_2^1(\Omega);$$

$$\Psi(t) = \langle Ma, b\rangle_{1,M}, \qquad F(t, x) = a_t - f(t, x) + La,$$

$$\Phi^\eta(t) = \varphi_2(t) - \frac{\eta}{2}\,\langle Ma_t, h^\eta\rangle_{1,M} + (f(t, x) - a_t, h^\eta),$$

2. The main result

In this section we prove the local existence and uniqueness theorem for Problem 1.

Theorem 2.1. *Let the operators M and L satisfy* I–IV *and η is a positive constant. Assume that*

(i) $f(t,x) \in C([0,T]; L^2(\Omega))$, $\beta(t,x) \in C^1([0,T]; W_2^{3/2}(\partial\Omega)$, $U_0(x) \in L^2(\Omega))$, $\varphi_i(t) \in C[0,T]$, $i = 1,2$;

(ii) $U_0(x)$, $\beta(t,x)$, $\omega(t,x)$, $\varphi_1(t)$, $\varphi_2(t)$ *are nonnegative functions and*

$$\int_\Omega h^\eta \,\mathrm{d}x \geq h_0 = \text{const} > 0, \qquad t \in [0,T];$$

(iii) *there exists a positive constant* $\alpha > 0$ *such that*

$$\Psi(t) \geq \alpha, \qquad t \in [0,T],$$

$$a(0,x) - U_0(x) \geq 0, \qquad x \in \Omega,$$

Then there exists T_0, $0 < T_0 \leq T$, such that Problem 1 has a solution $\big(u(t,x), k(t)\big) \in C^1([0,T_0]; W_2^2(\Omega)) \times C[0,T_0]$ and the solution is unique. Moreover, the coefficient $k(t)$ satisfies the estimate

$$|k(t)| \leq k_1 \tag{2.1}$$

with a positive constant k_1 for $t \in [0,T_0]$.

Proof. Following the idea in [9], we reduce Problem 1 to an equivalent inverse problem with a nonlinear operator equation for $k(t)$. Let us set $w(t,x) = a(t,x) - u(t,x)$. Then the pair $(w(t,x), k(t))$ is a solution of the problem

$$\begin{cases} w_t + \eta M w_t + k(t) M w + L w = F(t,x), & (t,x) \in Q_T; \\ (w + \eta M w)\big|_{t=0} = a(0,x) - U_0(x), \quad x \in \Omega; & w\big|_{\partial\Omega} = 0, \quad t \in (0,T); \end{cases} \tag{2.2}$$

$$\int_{\partial\Omega} \left\{ \eta \frac{\partial w_t}{\partial \nu} + k(t) \frac{\partial w}{\partial \nu} \right\} \omega \,\mathrm{d}s = (\varphi_1 + \Psi) k(t) - \varphi_2 + \eta \big\langle M a_t, h^\eta \big\rangle_{1,M}, \tag{2.3}$$

$t \in (0,T)$. By virtue of the integration by parts and (1.6), (2.3) we have

$$\begin{aligned} &(w_t + \eta M w_t, h^\eta)_0 + k(t)(M w, h^\eta) \\ &\quad = -\Big\{ (\varphi_1 + \Psi) k(t) - \varphi_2 + \eta \big\langle M a_t, h^\eta \big\rangle_{1,M} \Big\} - \frac{k(t)}{\eta} (w, h^\eta). \end{aligned} \tag{2.4}$$

Multiplying (2.2_1) by $h^\eta(t,x)$, integrating over Ω and substituting (2.4) into the resulting equation yield

$$k(t) \Big(\varphi_1(t) + \Psi(t) + \frac{1}{\eta}(w, h^\eta) \Big) = \Phi^\eta(t) - (L(a - w), h^\eta). \tag{2.5}$$

It is easily seen that Problem 1 has a unique solution if and only if the problem (2.2), (2.5) has a unique solution. Therefore it is sufficient to prove the assertion of the theorem for (2.2), (2.5).

We seak a solution $(w(t,x),k(t))$ by the iteration scheme $\{(w^i(t,x),k^i(t))\}_{i=0}^{\infty}$:

$$\begin{cases} w^i_t+\eta M w^i_t+k^{i-1}(t)Mw^i+Lw^i=F(t,x), \\ (w^i+\eta M w^i)\big|_{t=0}=a(0,x)-U_0(x), \qquad w^i\big|_{\partial\Omega}=0, \end{cases} \tag{2.6}$$

$$k^i(t)\Big(\varphi_1(t)+\Psi(t)+\frac{1}{\eta}(w^i,h^\eta)\Big)=\Phi^\eta(t)-\big(L(a-w^i),h^\eta\big) \tag{2.7}$$

for $i=1,2,3,\ldots$; $\;w^0(t,x)\equiv 0$. The initial approximation k^0 being a positive constant determined later.

We begin with estimating $k^i(t)$, $i=1,2,3,\ldots$. Suppose that for $i=1,2,3,\ldots$ there exists t_0^{i-1} $\big(0<t_0^{i-1}\le T\big)$ and a positive constant k_2^{i-1} satisfying the inequality

$$|k^{i-1}(t)|\le k_2^{i-1} \quad \text{for} \quad t\in[0,t_0^{i-1}]. \tag{2.8}$$

Multiplying (2.6_1) by Mw^i and integrating over Ω, we obtain

$$\|\nabla w^i(t)\|^2+\eta\|Mw^i(t)\|^2\le B_2\exp\Big(B_1T+\frac{2t}{\eta}k_2^{i-1}\Big), \qquad t\in[0,t_0^{i-1}]. \tag{2.9}$$

Here we use the Gronwall's lemma, the embedding theorem and the properties (1.2) and (1.4). The positive constants B_1, B_2 depend on T, η, m_0, m_1, λ, mes Ω $\|a(0,x)-U_0\|$, $\|F\|_{C([0,T];L^2(\Omega))}$. Applying the Friedrichs' inequality to (2.9) gives

$$\|w^i(t)\|\le KB_2^{1/2}\exp\Big(\frac{1}{2}\Big(B_1T+\frac{2t}{\eta}k_2^{i-1}\Big)\Big), \quad t\in[0,t_0^{i-1}]. \tag{2.10}$$

Now acting with the operator $(I+\eta M)^{-1}: W_2^{-1}(\Omega)\to \overset{\circ}{W}{}_2^1(\Omega)$ (I is the identity operator) on (2.6_1) and integrating over $(0,t)$, $0\le t\le t_0^{i-1}$, we arrive at the integral equation for w^i:

$$w^i=(I+\eta M)^{-1}(a(0,x)-U_0)+\int_0^t (I+\eta M)^{-1}\big(F-k^{i-1}(\tau)Mw^i-Lw^i\big)\,\mathrm{d}\tau.$$

Inserting this expression into the left side of (2.7) yields

$$k^i(t)\Big[\varphi_1(t)+\Psi(t)+\frac{1}{\eta}\big((I+\eta M)^{-1}(a(0,x)-U_0),h^\eta\big)$$
$$+\frac{1}{\eta}\Big(\int_0^t (I+\eta M)^{-1}\big(F-k^{i-1}(\tau)Mw^i-Lw^i\big)\,\mathrm{d}\tau,h^\eta\Big)\Big]=\Phi^\eta(t)-\big(L(a-w^i),h^\eta\big).$$

Let $i=1$. Since k^0 is a positive constant, the inequalities (1.2), (1.4), (2.9), (2.10) and the assumption (iii$'$) of Theorem 2.1 give

$$\frac{1}{\eta}\big((I+\eta M)^{-1}(a(0,x)-U_0),h^\eta\big)+\varphi_1(t)+\Psi(t)$$
$$+\frac{1}{\eta}\Big(\int_0^t (I+\eta M)^{-1}\big(F-k^0Mw^1-Lw^1\big)\,\mathrm{d}\tau,h^\eta\Big)\ge\alpha-t\big(B_3+k^0B_4\big)\ge\frac{\alpha}{2} \tag{2.11}$$

for $\forall t \in [0, t_0]$, $t_0 = \min\left\{\frac{\alpha}{2(B_3+k^0B_4)}, \ \frac{\eta}{2k^0}, T\right\}$ where positive constants B_3 and B_4 depends on m_0, m_1, λ, η, T, B_1, B_2, mesΩ, $\|\varphi_1\|_{C^1([0,T])}$, $\|\Psi\|_{C^1([0,T])}$, $\|F\|_{C([0,T];L^2(\Omega))}$, $\|h^\eta\|_{C([0,T];L^2(\Omega))}$. One can now conclude from (2.10)-(2.11) that $k^1(t)$ satisfies the inequality

$$|k^1(t)| \le \frac{2}{\alpha}\Big\{\overline{\Phi}^\eta + \lambda\Big[KB_2^{1/2}\exp\Big(\frac{B_1T+1}{2}\Big) + \max_{t\in[0,T]}\|a\|\Big]\Big\}\max_{t\in[0,T]}\|h^\eta\| \equiv B_5, \qquad (2.12)$$

for $t \in [0, t_0]$ Let us take $k^0 \le B_5$. Then the estimate (2.12) is valid for any $t \in [0, T_0]$, where

$$T_0 = \min\left\{\frac{\alpha}{2(B_3+B_4B_5)}, \ \frac{\eta}{2B_5}, \ T\right\}.$$

Next let $i = 2$. Since inequality (2.12) holds on $[0, T_0]$, it follows that w^2 satisfies (2.10) with $k_3^1 = B_6$. Hence the inequality (2.11) is valid on $[0, T_0]$ for w^2 and k^2, which implies

$$\left|k^i(t)\right| \le B_5 \quad t \in [0, T_0]. \qquad (2.13)$$

for $i = 2$. Repeating this procedure, one can deduce the estimate (2.13) and (2.8) with $k_2^{i-1} = B_5$ on $[0, T_0]$ for every $i = 1, 2, 3, \ldots$, which enables to derive the estimates

$$\eta\left\|w^i(t)\right\|_2^2 \le B_6, \quad \eta\left\|w_t^i(t)\right\|_2^2 \le B_7, \qquad t \in [0, T_0], \quad i = 1, 2, 3, \ldots \qquad (2.14)$$

from (2.8)-(2.10). The positive constants B_6 and B_7 depend on η, T_0, B_1, B_2, B_5, K, m_1, m_2, λ, $\max_{t\in[0,T]}\|F(t)\|$ but does not depend on i.

Now let $\tilde{k}^i(t) = k^{i+1}(t) - k^i(t)$, $\tilde{w}^i(t,x) = w^{i+1}(t,x) - w^i(t,x)$. From (2.7),(2.13),(2.14) it follows that

$$|\tilde{k}^i(t)| \le B_8\|\tilde{w}^i(t)\|, \qquad t \in [0, T_0], \qquad (2.15)$$

where the positive constant B_8 depends on η, α, k_1, λ, $\max_{t\in[0,T]}\|h^\eta\|$ but does not depend on i. The function $\tilde{w}^i$ is a solution of the problem

$$\begin{cases} \tilde{w}_t^i + \eta M\tilde{w}_t^i + k^i(t)M\tilde{w}^i + L\tilde{w}^i = -\tilde{k}^{i-1}(t)Mw^i, \\ \left(\tilde{w}^i + \eta M\tilde{w}^i\right)\big|_{t=0} = 0, \quad \tilde{w}^i\big|_{\partial\Omega} = 0. \end{cases} \qquad (2.16)$$

Multiplying (2.16_1) by $M\tilde{w}^i$ in terms of the inner product of $L^2(\Omega)$ and integrating by parts in the resulting equation, one can get by Cauchy's inequality, (1.2), (1.3), (2.13) and (2.14)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\langle\tilde{w}^i, M\tilde{w}^i\right\rangle_{1,M} + \eta\left\|M\tilde{w}^i\right\|^2\right) \le \left|\tilde{k}^{i-1}(t)\right|^2 + B_9\left(\left\langle\tilde{w}^i, M\tilde{w}^i\right\rangle_{1,M} + \eta\left\|M\tilde{w}^i\right\|^2\right) \quad (2.17)$$

where the constant B_9 depends on λ, B_5, B_6, m_1, m_3, η. Applying Gronwall's lemma to (2.17) yields

$$\left\langle\tilde{w}^i, M\tilde{w}^i\right\rangle_{1,M} + \eta\left\|M\tilde{w}^i\right\|^2 \le B_{10}\left(\int_0^t\left|\tilde{k}^{i-1}(\tau)\right|^2\mathrm{d}\tau\right)^{1/2}. \qquad (2.18)$$

The positive constant B_{10} depends on η, B_9, T_0, λ, $m_j (j = 1, 2, 3)$, mes Ω, but does not depend on i. Similarly one can derive the following estimate of $\bar{w}_t^i$ from (2.16_1) and (2.18).

$$\|\tilde{w}_t^i(t)\|_2 \le B_{11}\left[\left(\int_0^t\left|\tilde{k}^{i-1}(\tau)\right|^2\mathrm{d}\tau\right)^{1/2} + |\tilde{k}^{i-1}|\right], \qquad (2.19)$$

where the positive constant B_{11} depends on η, C_1, C_6, T_0, λ, $m_j (j = 1, 2, 3)$, mesΩ, but does not depend on i.

Let us introduce an equivalent norm in $C([0, T_0])$ as $\| \cdot \|_{\mu,T_0} = \max_{t\in[0,T_0]}\{e^{-\mu t}| \cdot |\}$ with a positive constant μ to be determined later. Then (2.15), (2.18) imply

$$\| \tilde{k}^i(t) \|_{\mu,T_0} \leq \frac{B_8 B_{10}}{\sqrt{2\mu}} \| \tilde{k}^{i-1}(t) \|_{\mu,T_0} \leq \left(\frac{B_8 B_{10}}{\sqrt{2\mu}}\right)^i \| \tilde{k}^0(t) \|_{\mu,T_0}.$$

The last inequality shows that there exists a limit $k(t)$ of the sequence $\{k^i(t)\}$ when μ satisfies the inequality $\mu > (B_8 B_{10})^2/2$.

This in turn provides the convergence of $\{w^i\}$ to a function $w(t, x)$ in the norm of $C([0, T_0]; W_2^2(\Omega))$ because of (2.19). Letting $i \to \infty$ in (2.6), (2.7), we see that the pair $(w(t, x), k(t))$ is the solution of the problem (2.2), (2.5). Besides, $w(t, x)$ satisfies the estimates (2.14). The estimates (2.1) follows from (2.14) immediately.

The uniqueness of the solution (w, k) follows from the inequalities for the difference $(\bar{w}, \bar{k}) = (w' - w'', k' - k'')$ of two solutions (w', k'), (w'', k'') of the problem (2.2), (2.5)

$$\|\bar{w}(t)\|_2 \leq B_{13} \left(\int_0^t \bar{k}(\tau)^2 \, d\tau\right)^{1/2}, \quad \| \bar{k}(t) \|_{\mu,T_0} \leq \frac{B_{13}}{\sqrt{2\mu}} \| \bar{k}(t) \|_{\mu,T_0}, \quad t \in [0, T_0].$$

The positive constant B_13 depends on η, T_0, B_1, B_3, B_6, $\max_{t\in[0,T]} \|F\|$, λ, K, m_1, m_2. If $\mu > B_{13}^2/2$, then $\bar{k}(t) \equiv 0$, and hence $\bar{w}(t, x) \equiv 0$ for all $t \in [0, T_0]$ and almost all $x \in \Omega$.

Список литературы

[1] Barenblatt G.I., Zheltov Iu.P., Kochina I.N. Basic concepts in the theory of seepage of homogeneous liquids in fissured blocks [strata] // Journal of Applied Mathematics and Mechanics. 1960. Vol. 24. P. 1286–1303.

[2] Chen P.J., Gurtin M.E. On a theory of heat conduction involving two temperatures // Z. Angew. Math. Phys. 1968. Vol. 19. P. 614–627.

[3] Korpusov M.O., Sveshnikov A.G. Blow-up of solutions of strongly nonlinear equations of pseudoparabolic type // J. Math. Sciences. 2008. Vol. 148. P. 1–142.

[4] Lorenzi A., Paparoni E. Identification problems for pseudoparabolic integrodifferential operator equations // J. Inver. Ill-Posed Probl. 1997. Vol. 5. P. 235–253.

[5] Mamayusupov M.Sh. The problem of determining coefficients of a pseudoparabolic equation // Studies in integro-differential equations / "Ilim". Frunze. 1983. No 16. P. 290–297.

[6] W. Rundell Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data // Appl. Anal. 1980. Vol. 10. P. 231–242.

[7] Lyubanova A.Sh. Identification of a constant coefficient in an elliptic equation // Appl. Anal. 2008. Vol. 87. P. 1121–1128.

[8] Li T.-T., White L.W. Total Flux (Nonlocal) Boundary Value Problems for Pseudoparabolic Equation // Appl. Anal. 1983. Vol. 16. P. 17–31.

[9] Prilepko A.I., Orlovsky D.G., Vasin I.A. Methods for solving inverse problems in mathematical physics. New York: Marcel Dekker,Inc., 2000. 708 p.