Approximation efficient algorithms with performance guarantees for some hard routing problems

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We make several observations on efficient approximation algorithms with proven guarantees for some discrete routing problems that are NP-hard in general case. One of the most popular problems of this kind is the Travelling Salesman Problem (TSP) [26]. The problem is MAX SNP-hard: existence of a polynomial approximation scheme for it yields $P = NP$.

Another problem considered is a problem of finding several edge-disjoint Hamiltonian circuits of extreme summary edge weight. The first mention of the problem, called m-Peripatetic Salesman Problem (m-PSP), in the literature came in [21]. The problem consists in finding edge-disjoint Hamiltonian circuits $H_i \subset E$, $i = \overline{1,m}$, in complete undirected n-vertex graph $G = (V, E)$ with weight functions $w_i : E \to R$, $i = \overline{1,m}$, such that their total weight $W_1(H_1) + \ldots + W_m(H_m)$ is minimal (or maximal), where it is denoted $W_i(H) = \sum_{e \in H} w_i(e)$, $i = \overline{1,m}$.

De Kort [9] shows that the problem of finding two edge-disjoint Hamiltonian circuits is NP-complete. This result implies that 2-PSP with identical weight functions is NP-hard both in maximization and minimization variants. The problem is also NP-hard for the case of different weight functions [3].

De Brey and Volgenant [8] identify several polynomially solvable cases of 2-PSP. De Kort [9, 10, 11] design and analyze lower and upper bounds for 2-PSP as possible ingredients of branch-and-bound algorithms. Duchenne, Laporte and Semet discass a polyhedral approach for solving m-PSP [12].

The problem on minimum we denote by m-PSP$_{min}$ and m-PSP$^d_{min}$(in the case common and different weight function correspondingly). Denotation for the problem on maximum is similar. We use also notation of kind 2-PSP$[1, q]$ if edge-weights possess the arbitrary values on interval $[1, q]$, and 2-PSP$\{r, q\}$ if weight function takes on two values r and q.

Recently we developed several polynomial algorithms with performance guarantees for solving problems 2-PSP and metric 2-PSP (in cases of common and different weight functions), 2-PSP$_{max}$ and m-PSP$_{max}$ on graphs in multidimensional Euclidean space. It is considered also some problems on random instances: the multi-index axial and planar assignment problems, Vehicle Routing Problems with restricted number of clients in each rout (k-VRP) for cases single and multiple depot.

1 Finding two edge disjoint Hamiltonian circuits

1.1 2-PSP$_{max}$: algorithms $A_{3/4}$ and $A_{7/9}$

A polynomial approximation algorithm $A_{3/4}$ with performance guarantee of 3/4 for solving 2-PSP$_{max}$ was presented in [1]. First of all a cubic (if n is even) of "almost" cubic (if n is odd) subgraph G_3 of G of maximum total edge weight is constructed by Gabow's algorithm [13]. Then G_3 is split into partial tour and 2-matching. Then the subgraphs are modified into two partial tours by regrouping their edges. The resulting partial tours can be fulfilled into two edge-disjoint Hamiltonian circuits by adding other edges. The weight of the solution is at least $(3/4)\,OPT$, where OPT is the optimal weight. The performance guarantee relies on the facts that the summary edge weight of G_3 is at least $(3/4)\,OPT$ and all the edges of G_3 were included into the solution. The running time of the algorithm is determined by the time complexity of finding G_3 in G and is bounded by $O(n^3)$ as described by Gabow [13].

In [19] it is presented a cubic time approximation algorithm $A_{7/9}$ for this problem with guaranteed ratio 7/9, the best known for today. The starting point of the algorithm is finding 4-regular subgraph $G_4 \subset G$ of maximum edge weight using Gabow's algorithm [13]. Then a couple of edge-disjoint specific tours are found in G_4 with great enough number of edges, following which these tours are transformed to the tours with total weight at least $(7/9)\,OPT$ and finally completed to edge-disjoint Hamiltonian cycles that correspond to approximate solution of the problem.

1.2 Metric 2-PSP$_{min}$: $A_{9/4}$ and A_2

It is supposed in this section that the triangle inequality holds: $w(i,j) \le w(i,k) + w(j,k)$ for each vertices $i,j,k \in V$. It is known that the problem 2-PSP$_{min}$ is NP-hard even in the metric case and does not admit any constant-factor approximation in the general case.

A cubic time approximation algorithm $A_{9/4}$ for solving Metric 2-PSP$_{min}$ was presented in [3]. Performance ratio of this algorithm asymptotically tends to 9/4.. Later the performance ratio of 9/4 for the problem was also announced in [7].

In [2] it is presented a 2-approximation algorithm A_2 for solving Metric 2-PSP$_{min}$. In due time for Metric TSP Christofides and Serdyukov well known 3/2-approximation algorithm constructed using transformation of shortest spanning tree to Hamiltonian circuit. In [2] Ageev and Pyatkin proceed from two edge-disjoint spanning trees (T_1^*, T_2^*) of minimum total weight that can be done in time $O(n^2 \log n)$ using the algorithm Roskind and Tarjan [22]. At Stage 1, transforming the couple (T_1^*, T_2^*) to another pair (T_1, T_2), satisfying $T_1^* \cup T_2^* = T_1 \cup T_2$, it is built first Hamiltonian cycle H_1,which is edge-disjoint with T_2 and has a weight at most $2W(T_1)$. At Stage 2, no changing edges between H_1 and T_2, the second Hamiltonian cycle H_2 is built. Meanwhile graphs $H_1 \cup T_1$ $H_2 \cup T_2$ are outer planar with outer faces H_1 H_2 correspondingly.

1.3 Metric 2-PSP$^d_{min}$

For this problem 12/5-approximation algorithm with the time complexity $O(n^3)$ was presented in [3]. Initially two approximate solutions H_1 and H_2 of TSP$_{\rm min}$ with weight functions w_1 and w_2 respectively are found by the 3/2-approximation Christofides-Serdyukov's algorithm. After that a second circuit H_2 is transformed in H_2' such that H_2' is edge-disjoint with H_1 and whose weight is at most twice the weight of H_1. Then roles of graphs H_1 and H_2 are exchanged and the pair (H_1, H_2') or (H_1', H_2) of minimum total weight is chosen as an approximate solution of the problem considered.

1.4 2-PSP$_{min}[1,q]$

The problem 2-PSP$_{min}[1,q]$ can be solved in $O(n^3)$-time with performance ratio $(4+q)/5$ [14]. A central place in proving this result belongs to the following (useful for construction and analysis algorithms) structure statement: in n-vertex 4-regular graph a pair of edge-disjoint partial tours with total number of edges at least 8n/5 can be found in quadratic time complexity.

1.5 2-PSP$_{min}\{1,2\}$

It is clear that 2-PSP$_{min}\{1,2\}$ is particular case of Metric problem.

In [7] performance ratio of about 1.37 was announced for this problem in assumption that performance ratio 7/6 holds for solution TSP$_{min}\{1,2\}$, found by algorithm presented in [24].

In [5] the following connection between problems on maximum and minimum is shown: let there be a polynomial ρ-approximation algorithm for the problem 2-PSP$_{max}(0,1)$. Then $(2-\rho)$-approximate solution for 2-PSP$_{min}(1,2)$ can be found in polynomial running-time. Thus approximate solutions with total weight of at most 5/4 and 11/9 times the optimal for 2-PSP$_{min}(1,2)$ can be found in $O(n^3)$ running-time using algorithms from [1] and [19] correspondingly.

Improved approximation ratio 6/5 results from the structure statement in previous section.

1.6 2-PSP$_{max}[1,q]$

In [16] a combined using of the 3/4-approximation algorithm for 2-PSP$_{max}$ and the $5/(q+4)$-approximation algorithm for 2-PSP$_{min}[1,q]$, that follows from [14], an improved approximation ratio $(3q+2)/(4q+1)$ for solving the problem 2-PSP$_{max}[1,q]$ is achieved. It means also the bound 8/9 for the problem with $q=2$.

1.7 2-PSP$^d_{min}(1,2)$

In [14] it is shown that a solution with performance ratio $(1+\rho'/2)$ can be obtain, where ρ' is the approximation guarantee for the problem TSP$_{min}(1,2)$. Using an algorithm from [6] with $\rho' = 8/7$ for solving TSP$_{min}(1,2)$ it is possible to find a feasible solution of 2-$PSP^d_{min}(1,2)$ with total weight of at most 11/7 of the optimal. Though the running time of the algorithm used is polynomial, it is very high: $O(n^{K+4})$, where the constant K in [6] is equal to 21. Thus, using an 11/9-approximation algorithm for $TSP(1,2)$ from [24], greater value of 29/18 implies, however the running-time of the algorithm is much smaller. In this case the time complexity is determined by one of the stages of the algorithm, where a minimum-weight cycle cover in G with edge-weights 1 and 2 is found. In [24] it is proposed, that it can be done in time $O(n^{2.5})$.

In [20] it is presented the 4/3-approximation algorithm with time complexity $O(n^4)$ for 2-PSP$^d_{min}(1,2)$. This result improves above mentioned performance guarantees 11/7 and 29/18. Algorithm is based on the ideas from [6], where 8/7-approximation algorithm for TSP$_{min}$ with edge weights 1 and 2 is announced. In particular, a perspective charge technique is used, applying for prove structure theorems in graph theory.

1.8 2-PSP$^d_{max}(1, 2)$

For the problem 2-PSP$^d_{max}(1, 2)$, the polynomial approximation algorithm constructed whose guaranteed exactness depends on approximation ratio $\rho \geq 1$ (known for the minimization version of the problem) as a function $\frac{11\rho-8}{18\rho-15}$. Then to known bounds 11/7, 7/5 and 4/3 of ρ for the problem 2-PSP$^d_{min}(1, 2)$, that were mentioned above, it is correspond the following estimations of performance ratios for considered problem 2-PSP$^d_{max}(1, 2)$: $\frac{65}{113} < \frac{37}{51} < \frac{20}{27}$ respectively [16].

2 Euclidean m-PSP$_{max}$

The problem TSP is called Euclidean if vertexes in graph correspond to points in Euclidean space R^k, and edge-weights equal to lengths of relative intervals.

It is known [Fekete&Barvinok] that Euclidean TSP$_{max}$ (ETSP$_{max}$) in space R^k is NP-hard when $k \geq 3$. (For $k = 2$ hardness status of ETSP$_{max}$ is open).

Nevertheless for ETSP$_{max}$ it does work asymptotically exact algorithm [23] with time complexity $O(n^3)$. The idea of solving the problem is to transform maximum weight matching (cycle cover) into Hamiltonian circuit by means of consecutive patching "near-parallel" matching edges (intervals) into cycles. Preliminarily maximum weight matching is divided into heavy and light edges. At first the heavy edges are patched, and then the light edges are used.

This idea was used for solving the problems of several edge-disjoint Hamiltonian circuits: 2-EPSP$_{max}$ [15] and m-EPSP$_{max}$ [4]. Noteworthily that the same maximum-weight matching intervals are used as "building material" or more exactly as "falsework" when in use successive constructing Hamiltonian cycles $H_1, \dots, H_m$.

In [4] for solving m-EPSP$_{max}$ an approximation algorithm with time complexity $O(n^3)$ and relative error $O\big((\frac{m}{n})^{\frac{2}{k+1}}\big)$ is constructed. So the algorithm is asymptotically exact under the condition $m = o(n)$ on the number of edge-disjoint Hamiltonian routs.

3 Multi-index Assignment Problem (MAP)

The MAP is NP-hard for the number of indexes at least three in axial and planar cases both [25].

In the case of **the of axial MAP**, n elements must be selected in the multi-dimensional matrix such that in every "cross-section" exactly one element is chosen. (The *"cross-section"* is such set of matrix elements when one index is fixed).

For the multi-index axial Assignment Problem on random instances, asymptotic optimality conditions were established for a quadratic-time algorithm based on the choice of minimal element in the current line of a special matrix formed from the initial matrix [3].

The three-index Planar Assignment Problem deals with selection of n^2 elements in a cubic matrix (c_{ijk}). Exactly one element in each line is chosen.
(*A line* is the set of n elements with two fixed indexes).

Conditions of asymptotic optimality were established for the m-layer three-index planar assignment problem on random input data when the number m of layers in the matrix (c_{ijk}) is at most $O(\ln n)$ [3] and $O(n^\theta)$, $0 < \theta < 1$ [18].

Note that m-layer three-index planar assignment problem on single-cyclic permutations coincides with the m-Peripatetic Salesman Problem.

4 k-VRP on random instances

k-VRP is a typical routing problem. There are applications in logistics, tracing on large-scale integrated circuits, the organization of drilling robot on board and etc. The problem consists in finding a family clients-disjoint routs of vehicles, when each client is served by exactly one vehicle, each of vehicle comes out of the specific vertex (the depot), visits (serves) no more then k vertices (clients) and moves back to the depot. The goal is to minimize the total length of all routs.

Some approximation algorithms for solving k-VRP on random data was constructed [17]. A probabilistic analysis was completed under the assumption that the distances between the vertexes of graph are independent random variables having the common distribution function like uniform, exponential, truncated normal and majorized type.

For k-VRP and multi-depot k-VRP it is obtained the bounds of the relative errors, the failure probabilities of the approximation algorithms and conditions of their asymptotic exactness on random initial data. It is shown in [17] that k-VRT on inputs UNIFORM(a_n, b_n) can be solved asymptotically exact if $k \geq \sqrt{2n}$ and $b_n/a_n = o(n/\ln n)$.

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References

[1] A.A. Ageev, A.E. Baburin, E.Kh. Gimadi. A 3/4–approximation algorithms for finding two disjoint Hamiltonian cycles of maximum weight // J. of Appl. and Ind. Math., 1(2). 2007. P. 142–147. Pleiades Publ. Ltd.

[2] A.A. Ageev, A.V. Pyatkin. A 2-approximation algorithm for the Metric 2-PSP // Discr. analyisis and Oper. res., 16(4). 2009. P. 3-20 (in Russian).

[3] A.E. Baburin, E.Kh. Gimadi, N.M. Korkishko. Approximation algorightms for finding two edge-disjoint Hamiltonian cycles of minimal total weight // Discr. analyisis and Oper. res., Ser 2. 11(1). 2004. P. 11–25 (in Russian).

[4] A.E. Baburin, E.Kh. Gimadi. On the Asymptotic Optimality of an Algorithm for Solving the Maximum m-PSP in a Multidimensional Euclidean Space // Proc. of the Steklov Institute of Mathematics, 2011, P. 3–16.

[5] A.E. Baburin, F.Della Croce, E.K. Gimadi, Y.V. Glazkov and V.Th. Paschos. Approximation algorithms for the 2-PSP with edge weights 1 and 2 // Discr. Appl. Math., 157(9), 2009, P. 1988–1992.

[6] Berman P., Karpinski M. 8/7–approximation algorithm for (1,2)-TSP // Proc. 17th ACM-SIAM SODA, 2006. P. 641–648.

[7] Croce F. D., Pashos V. Th., Calvo R. W. Approximating the 2-PSP // 7th Workshop on Modelling and Algorithms for Planning and Scheduling Problems MAPS 2005. (Siena, Italy, June 6-10). 2005. P. 114–116.

[8] M. J. D. De Brey, A. Volgenant, Well-solved cases of the 2-PSP // Optimization, 39(3). 1997. P. 275–293.

[9] De Kort J.,B. J.M. Lower bounds for symmetric K-PSP // Optimization, 22(1). 1991. P. 113–122.

[10] De Kort J. B. J. M. Upper bounds for the Symmetric 2-PSP // Optimization, 23(4). 1992. P. 357–367.

[11] De Kort J. B. J. M. A branch and bound algorithm for Symmetric 2-PSP // EJOR, 70. 1993. P. 229–243.

[12] Duchenne E., Laporte G., Semet F. Branch-and-Cut Algorithms for the Undirected m-PSP // EJOR, 162. 2005. P. 700-712.

[13] Gabow H.N. An efficient reduction technique for degree-constrained subgraph and bidirected network flow problems // Proc. 15th annual ACM symposium on theory of computing, (Boston, April 25-27, 1983). New York: ACM, 1983. P. 448–456.

[14] E.Kh. Gimadi, Yu.V. Glazkov, A.N. Glebov. Approximation algorithms for solving 2-PSP on complete graph with edge weight 1 and 2 // J. of Appl. and Ind. Math., 3(1). 2009. P. 46–60. Pleiades Publ. Ltd.

[15] E.Kh. Gimadi. Asymptotically Exact Algorithm for Finding One and Two Edge-Disjoint Traveling Salesman Routes of Maximal Weight in Euclidean Space // Proc. of the Steklov Institute of Mathematics, 2008 (transl. from Trudy Inst. Mat. Mekh. UrO RAN. 13 (1). 2008. P. 23–32).

[16] E.Kh. Gimadi, E.V. Ivonina. Approximation algorithms for maximum-weight 2-PSP in complete graph with restricted distances // Avtomatika i Telemekhanika. 2011 (submited in Russian).

[17] E.Kh. Gimadi, A.V. Shakhshneyder. Approximation algorithms with performance guarantees for routing problems on random inputs with restricted number of clients in each rout // Avtomatika i Telemekhanika. 2011 (submited in Russian).

[18] E.Kh. Gimadi, Yu.V. Glazkov. An asymptotically optimal algorithm for one modification of planar three-index assignment problem // J. of Appl. and Ind. Math., 1(4). 2007. P. 442–452. Pleiades Publ. Ltd.

[19] A.N. Glebov, D.Zh. Zambalaeva. A polynomial-time 7/9-approximation algorithm for two-travelling salesmen problem on maximum // Discr. analysis and Oper. Res., 18(4). 2011 (in Russian).

[20] A.N. Glebov, A.V. Gordeeva, D.Zh. Zambalaeva. Approximation algorithm for solving a problem on two travelling salesmen with different weight functions // Discr. analysis and Oper. Res., 18(5). 2011 (in Russian).

[21] J. Krarup. The peripatetic salesman and some related unsolved problems // Combinatorial programming: methods and applications (Proc. NATO Advanced Study Inst., Versailles, 1974). Dordrecht: Reidel, 1975. P. 173–178.

[22] J. Roskind and R.E. Tarjan, A note on Fiding minimum-cost edge-disjoint spanning trees // Math. Oper. Res., 10(4). 1985. P. 701–708.

[23] Serdukov A.I. An Asymptoticaly Optimal Algorithm for Solving Max Euclidean TSP // Metodi Celochislennoi Optimizacii (Upravliaemie Sistemi), Novosibirsk. 27. 1987. P. 79–87 (in Russian).

[24] Papadimitriu C. H., Yannakakis M. The traveling salesman problem with distance One and Two // Math. Oper. Res., 18(1). 1993. P. 1–11.

[25] Spieksma F.C.R. Multi index assignment problems: complexity, approximation, applications // Nonlinear assignment problems, algorithms and applications. 2000. Kluwer Acad. Publ., 2000. P. 1–12.

[26] The Traveling Salesman Problem and its Variations, Gutin G., Punnen A.P. (eds.) Kluver Academic Publishers, Dordrecht / Boston / London. 2002. 830 p.