CONVEXITY AND INEQUALITY

Kutateladze S.S.
Sobolev Institute of Mathematics, Novosibirsk, Russia
sskut@math.nsc.ru

Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

Convexity reigns in the federation of geometry, optimization, and functional analysis. Convexity feeds generation, separation, calculus, and approximation. Generation appears as duality; separation, as optimality; calculus, as representation; and approximation, as stability [1].

This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma [2]. Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators.

Through Ages

Mathematics resembles linguistics sometimes and pays tribute to etymology, hence, history. Today's convexity is a centenarian, and abstract convexity is much younger.

Convexity traces back to the idea of a solid figure in plane geometry. Book I of Euclid's *Elements* [3] reads:

Definition 13. A boundary is that which is an extremity of anything.

Definition 14. A figure is that which is contained by any boundary or boundaries.

Narrating solid geometry in Book XI, Euclid travelled from solid to surface:

Definition 1. A solid is that which has length, breadth, and depth.

Definition 2. An extremity of a solid is a surface.

Definition 9. Similar solid figures *are those contained by similar planes equal in multitude.*

Definition 10. Equal and similar solid figures *are those contained by similar planes equal in multitude and magnitude.*

Convexity and inequality stem from the remote ages [5]–[7]. However, as the acclaimed pioneers who propounded these ideas and anticipated their significance for the future we must rank the three polymaths:

Joseph-Louis Lagrange (January 25, 1736–April 10, 1813)
Jean-Baptiste Joseph Fourier (March 21, 1768–May 16, 1830)
Hermann Minkowski (June 22, 1864–January 12, 1909)

Joseph Lagrange (1736–1813)

In both research and exposition, he totally reversed the methods of his predecessors. They had proceeded in their exposition from special cases, by a species of induction; his eye was always directed to the highest and most general points of view; and it was by his suppression of details and neglect of minor, unimportant considerations that he swept the whole field of analysis with a generality of insight and power never excelled, adding to his originality and profundity a *conciseness, elegance, and lucidity which have made him the model of mathematical writers.* (Thomas J. McCormack [8])

Joseph Fourier (1768–1830)

He [Fourier] himself was neglected for his work on inequalities, what he called "Analyse indéterminée." Darboux considered that he gave the subject an exaggerated importance and did not publish the papers on this question in his edition of the scientific works of Fourier. Had they been published, linear programming and convex analysis would be included in the heritage of Fourier. (Jean-Pierre Kahane [9])

Hermann Minkowski (1864–1909)

Since my student years Minkowski was my best, most dependable friend who supported me with all the depth and loyalty that was so characteristic of him. Our science, which we loved above all else, brought us together; it seemed to us a garden full of flowers:.. In it, we enjoyed looking for hidden pathways and discovered many a newperspective that appealed to our sense of beauty and when one of us showed it to the other and we marvelled over it together, our joy was complete. He was for me a rare gift from heaven.... and I must be grateful to have possessed that gift for so long. Now death has suddenly torn him from our midst. However, what death cannot take away is his noble image in our hearts and the knowledge that his spirit in us continue to be active. (David Hilbert [10])

Convexity as Abstraction

Stretching a rope taut between two stakes produces a closed straight line segment, the continuum in modern parlance. Rope-stretching raised the problem of measuring the continuum. The continuum hypothesis of set theory is the shadow of the ancient problem of harpedonaptae. Rope-stretching independent of the position of stakes is uniform with respect to direction in space. The mental experiment of uniform rope-stretching yields a compact convex figure. ideas of experimental science and intrinsic to the geometric outlook.

Convexity has found solid grounds in set theory. The Cantor paradise became an official residence of convexity. Abstraction becomes an axiom of set theory. The abstraction axiom enables us to reincarnate a property, in other words, to collect and to comprehend. The union of convexity and abstraction was inevitable. This yields abstract convexity [11]–[13].

Environment for Convexity

Let $\overline{E}$ be a vector lattice E with the adjoint top $\top := +\infty$ and bottom $\bot := -\infty$. Assume further that H is some subset of E that is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or *H-convex* provided that U is the *H-support set*

$$U_p^H := \{h \in H : h \leq p\}$$

of some element p of $\overline{E}$. Limiting finite subsets of H-convex sets yields analogs of polyhedra.

An element $p \in \overline{E}$ is *H-convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The proper H-convex elements fill the cone $\mathscr{C}(H, \overline{E})$.

The *Minkowski duality* $\varphi : p \mapsto U_p^H$ enables us to study convex elements and sets simultaneously.

Lyapunov's Convexity Theorem

The celebrated Lyapunov Convexity Theorem had raised the problem of describing the compact convex sets in finite-dimensional real spaces which serve as the ranges of diffuse measures. These compacta are known in the modern geometrical literature as *zonoids*. Among zonoids we distinguish the Minkowski sums of finitely many straight line segments. These sets, called *zonotopes*, fill a convex cone in the space of compact convex sets, and the cone of zonotopes is dense in the closed cone of all zonoids. The first description of the ranges of diffuse vector measures in the Lyapunov Convexity Theorem was firstly found by Chuĭkina practically in the modern terms (see[14]). Soon after that her result was somewhat supplemented and simplified by Glivenko in [15]. The zonotopes of the present epoch were called *parallelohedra* those days.

Zonoids

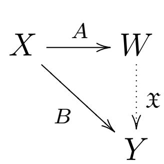
The significant further progress in studying the ranges of diffuse vector measures belong to Reshetnyak and Zalgaller who described zonoids as the results of mixing the linear elements of a rectifiable curve in a finite-dimensional space in 1954 (see [16]). In this same paper they suggested a new prove of the Lyapunov Convexity Theorem and demonstrated that zonotopes are precisely those convex polyhedra whose two-dimensional faces have centers of symmetry. Unfortunately, these results remained practically unnoticed in the West. Analogous results were obtained by Bolker only fifteen years later in 1969 (see [17], [18]).

Environment for Inequality

Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

Kantorovich's Theorem

Find $\mathfrak{X}$ satisfying



(1): $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.

(2): *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[1]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

The Farkas Alternative

Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.

Then one and only one of the following holds:

(1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leq b$ and*

$$b'Bx > 0, bA_1x \leq 0, \dots, bA_Nx \leq 0.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^N \alpha_k A_k$.*

Hidden Dominance

Lemma 1. *Let X be a vector space over some subfield R of the reals $\mathbb{R}$. Assume that f and g are R-linear functionals on X; in symbols, $f, g \in X^\# := L(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \{f \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha \in \mathbb{R}_+$ satisfying $g = \alpha f$.

Proof. Sufficiency is obvious.

Necessity: The case of $f = 0$ is trivial. If $f \neq 0$ then there is some $x \in X$ such that $f(x) \in \mathbb{R}$ and $f(x) > 0$. Denote the image $f(X)$ of X under f by R_0. Put $h := g \circ f^{-1}$, i .e. $h \in R_0^\#$ is the only solution for $h \circ f = g$. By hypothesis, h is a positive R-linear functional on R_0. By the Bigard Theorem [24, p. 108] h can be extended to a positive homomorphism $\bar{h} : \mathbb{R} \to \mathbb{R}$, since $R_0 - \mathbb{R}_+ = \mathbb{R}_+ - R_0 = \mathbb{R}$. Each positive automorphism of $\mathbb{R}$ is multiplication by a positive real. As the sought α we may take $\bar{h}(1)$.

The proof of Lemma 1 is complete.

Explicit Dominance

Lemma 2. *Let X be an $\mathbb{R}$-seminormed vector space over some subfield R of $\mathbb{R}$. Assume that $f_1, \dots, f_N$ and g are bounded R-linear functionals on X; in symbols, $f_1, \dots, f_N, g \in X^* := L^{(m)}(X, \mathbb{R})$.*

For the inclusion

$$\{g \leq 0\} \supset \bigcap_{k=1}^N \{f_k \leq 0\}$$

to hold it is necessary and sufficient that there be $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying

$$g = \sum_{k=1}^N \alpha_k f_k.$$

[1]Cp. [24, p. 51].

Boolean Modeling

Cohen's final solution of the problem of the cardinality of the continuum within ZFC gave rise to the Boolean valued models by Scott, Solovay, and Vopěnka.[2]

Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.[3]

Scott forecasted in 1969:[4]

We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

In 2009 Scott wrote:[5]

At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

Let $\mathbb{B}$ be a complete Boolean algebra. Given an ordinal α, put

$$V_\alpha^{(\mathbb{B})} := \{x \mid (\exists \beta \in \alpha)\ x : \mathrm{dom}(x) \to \mathbb{B}\ \&\ \mathrm{dom}(x) \subset V_\beta^{(\mathbb{B})}\}.$$

The *Boolean valued universe* $\mathbb{V}^{(\mathbb{B})}$ is

$$\mathbb{V}^{(\mathbb{B})} := \bigcup_{\alpha \in \mathrm{On}} V_\alpha^{(\mathbb{B})},$$

with On the class of all ordinals.

The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

Descending and Ascending

Given φ, a formula of ZFC, and y, a member of $\mathbb{V}^{\mathbb{B}}$; put $A_\varphi := A_{\varphi(\cdot,\ y)} := \{x \mid \varphi(x,\ y)\}$.

The *descent* $A_\varphi{\downarrow}$ of a class A_φ is

$$A_\varphi{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![\varphi(t,\ y)]\!] = \mathbb{1}\}.$$

If $t \in A_\varphi{\downarrow}$, then it is said that t *satisfies* $\varphi(\cdot,\ y)$ *inside* $\mathbb{V}^{(\mathbb{B})}$.

The *descent* $x{\downarrow}$ of $x \in \mathbb{V}^{(\mathbb{B})}$ is defined as

$$x{\downarrow} := \{t \mid t \in \mathbb{V}^{(\mathbb{B})}\ \&\ [\![t \in x]\!] = \mathbb{1}\},$$

i.e. $x{\downarrow} = A_{\cdot \in x}{\downarrow}$. The class $x{\downarrow}$ is a set.

If x is a nonempty set inside $\mathbb{V}^{(\mathbb{B})}$ then

$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

The *ascent* functor acts in the opposite direction.

There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

$$[\![\mathscr{R} \text{ is the reals}]\!] = \mathbb{1}.$$

[2]Cp. [19].

[3]Cp. [20].

[4]Cp. [21].

[5]Letter of April 29, 2009 to S. S. Kutateladze.

Let $\mathscr{R}{\downarrow}$ be the descent of the carrier $|\mathscr{R}|$ of the algebraic system $\mathscr{R} := (|\mathscr{R}|, +, \cdot, 0, 1, \leq)$ inside $\mathbb{V}^{(\mathbb{B})}$.

Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1} \ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

Gordon Theorem.[6] *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

We will proceed by Boolean valued analysis:

Theorem 1. *Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$.*

The following are equivalent:

(1) *Given $b \in \mathbb{B}$, the operator inequality $bBx \leq 0$ is a consequence of the simultaneous linear operator inequalities $bA_1x \leq 0, \dots, bA_Nx \leq 0$, i.e.,*

$$\{bB \leq 0\} \supset \{bA_1 \leq 0\} \cap \dots \cap \{bA_N \leq 0\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ such that*

$$B = \sum_{k=1}^{N} \alpha_k A_k;$$

i.e., B lies in the operator convex conic hull of $A_1, \dots, A_N$.

Proof. (2)→ (1): If $B = \sum_{k=1}^{N} \alpha_k A_k$ for some positive $\alpha_1, \dots, \alpha_N$ in $\mathrm{Orth}(m(Y))$ while $bA_kx \leq 0$ for $b \in \mathbb{B}$ and $x \in X$, then

$$bBx = b\sum_{k=1}^{N} \alpha_k A_k x = \sum_{k=1}^{N} \alpha_k bA_k x \leq 0$$

since orthomorphisms commute and projections are orthomorphisms of $m(Y)$.

(1)→ (2): Consider the separated Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ over the base $\mathbb{B}$ of Y. By the Gordon Theorem the ascent $Y{\uparrow}$ of Y is $\mathscr{R}$, the reals inside $\mathbb{V}^{(\mathbb{B})}$.

Using the canonical embedding, we see that $X^{\wedge}$ is an $\mathscr{R}$-seminormed vector space over the standard name $\mathbb{R}^{\wedge}$ of the reals $\mathbb{R}$.

Moreover, $\mathbb{R}^{\wedge}$ is a subfield and sublattice of $\mathscr{R} = Y{\uparrow}$ inside $\mathbb{V}^{(\mathbb{B})}$.

(1)→ (2):

Put $f_k := A_k{\uparrow}$ for all $k := 1, \dots, N$ and $g := B{\uparrow}$. Clearly, all $f_1, \dots, f_N, g$ belong to $(X^{\wedge})^*$ inside $\mathbb{V}^{\mathbb{B}}$.

Define the finite sequence

$$f : \{1, \dots, N\}^{\wedge} \to (X^{\wedge})^*$$

as the ascent of $(f_1, \dots, f_N)$. In other words, the truth values are as follows:

$$[\![f_{k^{\wedge}}(x^{\wedge}) = A_k x]\!] = \mathbb{1}, \quad [\![g(x^{\wedge}) = Bx]\!] = \mathbb{1}$$

for all $x \in X$ and $k := 1, \dots, N$.

(1)→ (2):

[6] Cp. [19, p. 349].

Put

$$b := [\![A_1x \le 0^\wedge]\!] \wedge \dots \wedge [\![A_Nx \le 0^\wedge]\!].$$

Then $bA_kx \le 0$ for all $k := 1, \dots, N$ and $bBx \le 0$ by (1).

Therefore,

$$[\![A_1x \le 0^\wedge]\!] \wedge \dots \wedge [\![A_Nx \le 0^\wedge]\!] \le [\![Bx \le 0^\wedge]\!].$$

In other words,

$$[\![(\forall k := 1^\wedge, \dots, N^\wedge) f_k(x^\wedge) \le 0^\wedge]\!] = \bigwedge_{k:=1,\dots,N} [\![f_{k^\wedge}(x^\wedge) \le 0^\wedge]\!] \le [\![g(x^\wedge) \le 0^\wedge]\!].$$

(1)→ (2):

By Lemma 2 inside $\mathbb{V}^{(\mathbb{B})}$ and the maximum principle of Boolean valued analysis, there is a finite sequence $\alpha : \{1^\wedge, \dots, N^\wedge\} \to \mathscr{R}_+$ inside $\mathbb{V}^{(\mathbb{B})}$ satisfying

$$[\![(\forall x \in X^\wedge)\ g(x) = \sum_{k=1^\wedge}^{N^\wedge} \alpha(k) f_k(x)]\!] = \mathbb{1}.$$

Put $\alpha_k := \alpha(k^\wedge) \in \mathscr{R}_+\!\downarrow$ for $k := 1, \dots, N$.

Multiplication by an element in $\mathscr{R}\!\downarrow$ is an orthomorphism of $m(Y)$. Moreover,

$$B = \sum_{k=1}^{N} \alpha_k A_k,$$

which completes the proof.

Absence of Dominance

Lemma 1, describing the consequences of a single inequality, does not restrict the class of functionals under consideration.

The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general.

The inclusion $\{f = 0\} \subset \{g \le 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{Q}$ and the identity automorphism of $\mathbb{Q}$.

Theorem 2. *Take A and B in $L(X, Y)$. The following are equivalent:*

(1) $(\exists \alpha \in \operatorname{Orth}(m(Y)))\ B = \alpha A$;

(2) *There is a projection $\varkappa \in \mathbb{B}$ such that*

$$\{\varkappa bB \le 0\} \supset \{\varkappa bA \le 0\}; \quad \{\neg\varkappa bB \le 0\} \supset \{\neg\varkappa bA \ge 0\}$$

for all $b \in \mathbb{B}$.[7]

Proof. Boolean valued analysis reduces the claim to the scalar case. Applying Lemma 1 twice and writing down the truth values, complete the proof.

[7] As usual, $\neg\varkappa := \mathbb{1} - \varkappa$.

Inexact Data

Let X be a vector lattice. An *interval operator* $\mathbf{T}$ from X to Y is an order interval $[\underline{T}, \overline{T}]$ in $L^{(r)}(X,Y)$, with $\underline{T} \le \overline{T}$.[8]

The interval equation $\mathbf{B} = \mathfrak{X}\mathbf{A}$ has a *weak interval solution* provided that $(\exists \mathfrak{X})(\exists A \in \mathbf{A})(\exists \mathbf{B} \in \mathbf{B})\ \mathbf{B} = \mathfrak{X}\mathbf{A}$.

Given an interval operator $\mathbf{T}$ and $x \in X$, put

$$P_{\mathbf{T}}(x) = \overline{T}x_+ - \underline{T}x_-.$$

Call $\mathbf{T}$ *adapted* in case $\overline{T} - \underline{T}$ is the sum of finitely many disjoint addends.

Put $\sim (x) := -x$ for all $x \in X$.

Theorem 3. *Let X be a vector lattice, and let Y be a Kantorovich space. Assume that $\mathbf{A}_1, \dots, \mathbf{A}_N$ are adapted interval operators and $\mathbf{B}$ is an arbitrary interval operator in the space of order bounded operators $L^{(r)}(X,Y)$.*

The following are equivalent:

(1) *The interval equation*

$$\mathbf{B} = \sum_{k=1}^{N} \alpha_k \mathbf{A}_k$$

has a weak interval solution $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(Y)_+$.

(2) *For all $b \in \mathbb{B}$ we have*

$$\{b\mathfrak{B} \ge 0\} \supset \{b\mathfrak{A}_1^{\sim} \le 0\} \cap \dots \cap \{b\mathfrak{A}_N^{\sim} \le 0\},$$

where $\mathfrak{A}_k^{\sim} := P_{\mathbf{A}_k} \circ \sim$ for $k := 1, \dots, N$ and $\mathfrak{B} := P_{\mathbf{B}}$.

Inhomogeneous Inequalities

Theorem 4. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X,Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \le bu_1, \dots, bA_Nx \le bu_N$, i.e.,*

$$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \dots \cap \{bA_N \le bu_N\}.$$

(2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Theorem 5.[9] *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A \in L^{(m)}(X, Y^s)$, $B \in L^{(m)}(X, Y^t)$, $u \in Y^s$, and $v \in Y^t$, where s and t are some naturals.*

The following are equivalent:

(1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent inhomogeneous inequality $bAx \le bu$, i.e., $\{bB \le bv\} \supset \{bA \le bu\}$.*

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[8] Cp. [22]

[9] Cp. [23].

(2) *There is some $s \times t$ matrix with entries positive orthomorphisms of $m(Y)$ such that $B = \mathfrak{X}A$ and $\mathfrak{X}u \le v$ for the corresponding linear operator $\mathfrak{X} \in L_+(Y^s, Y^t)$.*

Theorem 6. *Let X be a Y-seminormed complex vector space, with Y a Kantorovich space. Assume given some $u_1, \dots, u_N, v \in Y$ and dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y_{\mathbb{C}})$ from X into the complexification $Y_{\mathbb{C}} := Y \otimes iY$ of Y.*[10] *Assume further that the inhomogeneous simultaneous inequalities $|A_1x| \le u_1, \dots, |A_Nx| \le u_N$ are consistent. Then the following are equivalent:*

(1) *$\{b|B(\cdot)| \le bv\} \supset \{b|A_1(\cdot)| \le bu_1\} \cap \dots \cap \{b|A_N(\cdot)| \le bu_N\}$ for all $b \in \mathbb{B}$.*

(2) *There are complex orthomorphisms $c_1, \dots, c_N \in \operatorname{Orth}(m(Y)_{\mathbb{C}})$ satisfying*

$$B = \sum_{k=1}^{N} c_k A_k; \quad v \ge \sum_{k=1}^{N} |c_k| u_k.$$

Lemma 3. *Let X be a real vector space. Assume that $p_1, \dots, p_N \in \operatorname{PSub}(X) := \operatorname{PSub}(X, \mathbb{R})$ and $p \in \operatorname{Sub}(X)$. Assume further that $v, u_1, \dots, u_N \in \mathbb{R}$ make consistent the simultaneous sublinear inequalities $p_k(x) \le u_k$, with $k := 1, \dots, N$.*

The following are equivalent:

(1) *$\{p \ge v\} \supset \bigcap\limits_{k=1}^{N} \{p_k \le u_k\}$;*

(2) *there are $\alpha_1, \dots, \alpha_N \in \mathbb{R}_+$ satisfying*

$$(\forall x \in X)\ p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \ge 0, \quad \sum_{k=1}^{N} \alpha_k u_k \le -v.$$

Proof. (2) $\to$ (1): If x is a solution to the simultaneous inhomogeneous inequalities $p_k(x) \le u_k$ with $k := 1, \dots, N$, then

$$0 \le p(x) + \sum_{k=1}^{N} \alpha_k p_k(x) \le p(x) + \sum_{k=1}^{N} \alpha_k u_k(x) \le p(x) - v.$$

(1) $\to$ (2): Given $(x, t) \in X \times \mathbb{R}$, put $\bar{p}_k(x, t) := p_k(x) - tu_k$, $\bar{p}(x, t) := p(x) - tv$ and $\tau(x, t) := -t$. Clearly, $\tau, \bar{p}_1, \dots, \bar{p}_N \in \operatorname{PSub}(X \times \mathbb{R})$ and $\bar{p} \in \operatorname{Sub}(X \times \mathbb{R})$. Take

$$(x, t) \in \{\tau \le 0\} \cap \bigcap_{k=1}^{N} \{\bar{p}_k \le 0\}.$$

If, moreover, $t > 0$; then $u_k \ge p_k(x/t)$ for $k := 1, \dots, N$ and so $p(x/t) \le v$ by hypothesis. In other words $(x, t) \in \{\bar{p} \le 0\}$. If $t = 0$ then take some solution $\bar{x}$ of the simultaneous inhomogeneous polyhedral inequalities under study.

Since $x \in K := \bigcap_{k=1}^{N} \{p_k \le 0\}$; therefore, $p_k(\bar{x} + x) \le p(x) + p_k(x) \le u_k$ for all $k := 1, \dots, N$. Hence, $p(\bar{x} + x) \ge v$ by hypothesis. So the sublinear functional p is bounded below on the cone K. Consequently, p assumes only positive values on K. In other words, $(x, 0) \in \{\bar{p} \le 0\}$. Thus $\{\bar{p} \ge 0\} \supset \bigcap_{k=1}^{N} \{\bar{p}_k \le 0\}$ and by Lemma 2.2. of [1] there are positive reals $\alpha_1, \dots, \alpha_N, \beta$ such that for all $(x, t) \in X \times \mathbb{R}$ we have $\bar{g}(x) + \beta\tau(x) + \sum\limits_{k=1}^{N} \alpha_k \bar{p}_k(x) \ge 0$. Clearly, the so-obtained parameters $\alpha_1, \dots, \alpha_N$ are what we sought for. The proof of Lemma 3 is complete.

[10] Cp. [19, p. 338].

Theorem 7. *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given are some dominated polyhedral sublinear operators $P_1, \dots, P_N \in \mathrm{PSub}^{(m)}(X,Y)$ and a dominated sublinear operator $P \in \mathrm{Sub}^{(m)}(X,Y)$. Assume further that $u_1, \dots, u_N, v \in Y$ make consistent the simultaneous inhomogeneous inequalities*

$$P_1(x) \le u_1, \dots, P_N(x) \le u_N.$$

The following are equivalent:

(1) *for all $b \in \mathbb{B}$ the inhomogeneous sublinear operator inequality $bP(x) \ge bv$ is a consequence of the simultaneous inhomogeneous sublinear operator inequalities $bP_1(x) \le bu_1, \dots, bP_N(x) \le bu_N$, i.e.,*

$$\{bP \ge bv\} \supset \{bP_1 \le bu_1\} \cap \dots \cap \{bP_N \le bu_N\};$$

(2) *there are positive $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \ge 0, \quad \sum_{k=1}^{N} \alpha_k u_k \le -v.$$

Lagrange's Principle. *The finite value of the constrained problem*

$$P_1(x) \le u_1, \dots, P_N(x) \le u_N, \quad P(x) \to \inf$$

is the value of the unconstrained problem for an appropriate Lagrangian without any constraint qualification other that polyhedrality.

The Slater condition allows us to eliminate polyhedrality as well as considering a unique target space. This is available in a practically unrestricted generality [24].

About the new trends relevant to the Farkas Lemma see [25]–[29].

Freedom and Inequality

Convexity is the theory of linear inequalities in disguise, tailored by set theory with a plentitude of bizarre visualizations of the figments of intuition.

Abstraction is the freedom of generalization. Freedom is the loftiest ideal and idea of man, but it is demanding, limited, and vexing. So is abstraction. So are its instances in convexity, hence, in simultaneous inequalities. Convexity and inequality supersede linearity because there are inequalities other than interpretations of simultaneous equalities.

Many promising opportunities are open up to modeling the powerful habits of reasoning and verification.

Inequality is the first and foremost phenomenon of being. Equality is second historically, linguistics notwithstanding.

Freedom presumes liberty and equality. Inequality paves way to freedom.

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